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Asymptotics of the ϕ_1^4 Measure in the Sharp Interface Limit

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Abstract

We consider the ϕ_1^4 measure in an interval of length ℓ, defined by a symmetric double-well potential W and inverse temperature β. Our results concern its asymptotic behavior in the joint limit $\beta, \ell \to \infty$, both in the subcritical regime $\ell \ll \mathrm{e}^{\beta C_W}$ and in the supercritical regime $\ell \gg \mathrm{e}^{\beta C_W}$, where C_W denotes the surface tension. In the former case, in which the measure concentrates on the pure phases, we prove the corresponding large deviation principle. The associated rate function is the Modica–Mortola functional modified to take into account the entropy of the locations of the interfaces. Furthermore, we provide the sharp asymptotics of the probability of having a given number of transitions between the two pure phases. In the supercritical regime, the measure no longer concentrates and we show that the interfaces are asymptotically distributed according to a Poisson point process.

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1. Introduction

The van der Waals' theory of phase transition [39] is based on the functional

$$\mathcal{F}(\phi) = \int \mathrm{d}x \left[\frac{1}{2} |\nabla \phi(x)|^2 + W(\phi(x)) \right],$$

where the scalar field ϕ represents the local order parameter and W is a smooth, symmetric, double well potential whose minimum value, chosen to be zero, is attained at ± 1, interpreted as the pure phases of the system. We assume that $W''(\pm 1) > 0$. After the celebrated Modica–Mortola result [28], the so-called gradient theory of phase transitions essentially amounts to the analysis of the sharp interface limit of the free energy functional $\mathcal{F}$, see [1] for a review. In the sequel, we restrict the discussion to the case $x \in [0, \ell]$ with free boundary conditions at the endpoints and denote by $\mathcal{F}_\ell$ the corresponding free energy functional.

The above variational formulation of phase transitions does not take into account the microscopic fluctuations, which are relevant in various phenomena. At the mesoscopic level, the effect of fluctuations can be encoded by considering the probability measure, on the space of order parameter profiles, informally given by

$$\mathrm{d}\mu_{\beta,\ell} \propto \mathcal{D}\phi \; \exp\{-\beta \mathcal{F}_\ell(\phi)\}. \tag{1.1}$$

Inspired by the paradigmatic case $W(\phi) = \frac{1}{2}(\phi^2 - 1)^2$, we refer to $\mu_{\beta,\ell}$ as the ϕ_1^4 measure, where the subscript one stands for the space dimension and the superscript refers to the growth of W. In general, the probability $\mu_{\beta,\ell}$ corresponds to the Euclidean version of the quantum anharmonic oscillator and it has been extensively analyzed since it exhibits an interesting behavior in a simple setting, see [38] and references therein.

Our present purpose is to analyze the behavior of the probability $\mu_{\beta,\ell}$ in the joint limit $\beta, \ell \to \infty$. The relevant length scale is $\bar{\ell} \approx \exp\{\beta C_W\}$, where C_W is the surface tension in the Van der Waals theory. Indeed, for $\ell \ll \bar{\ell}$ the typical behavior of the system is described by the pure phases $\phi_\pm = \pm 1$. In particular, due to the symmetry of the boundary conditions, in this regime $\mu_{\beta,\ell}$ converges to the convex combination of Dirac measures $\frac{1}{2}\delta_{\phi_-} + \frac{1}{2}\delta_{\phi_+}$. A relevant question is then the analysis of fluctuations. When ℓ grows sub-exponentially in β these are captured by the Modica–Mortola functional. On the other hand, as discussed in [30], when ℓ grows exponentially, the additional entropy induced by the large system size in (1.1) cannot be neglected, so that the probability of a profile with n interfaces is of the order $\ell^n \mathrm{e}^{-n\beta C_W}$. We here complete this picture by showing that the probability $\mu_{\beta,\ell}$, when suitably rescaled, satisfies a large deviation principle with a modified rate functional that takes into account the additional entropic contribution.

Within the regime $\ell \ll \bar{\ell}$, we also refine the result of [30] by obtaining the sharp asymptotics of the $\mu_{\beta,\ell}$-probability of suitable neighborhoods of profiles with n interfaces. Our approach leverages the fact that the finite-dimensional distributions of the ϕ_1^4 measure can be expressed in terms of the semigroup generated by the quantum anharmonic oscillator. It relies on precise semiclassical estimates of the spectral properties of this operator (see the outline of proofs at the end of Section 2). In particular, we derive the sharp asymptotic behavior of the characteristic length scale $\bar{\ell}$, which we relate to the spectral gap in the semiclassical limit, showing that $\bar{\ell} \propto \beta^{-1/2} \exp \beta C_W$.

By contrast, in the regime $\ell \gg \bar{\ell}$, there is enough entropy so that the $\mu_{\beta,\ell}$-probability of having at least one interface does not vanish. Accordingly, we show that the limiting location of the interfaces is distributed according to a Poisson point process. We refer to [10,34] for the analogous statement in the context of one-dimensional lattice models.

We remark that the sharp asymptotic analysis of the probability measure $\mu_{\beta,\ell}$ presented here is a preliminary step for getting sharp estimates on the metastable behavior, in the joint limit $\beta, \ell \to \infty$, of the stochastic Allen-Cahn equation, whose unique stationary distribution is $\mu_{\beta,\ell}$. While for ℓ fixed and $\beta \to \infty$ the corresponding Eyring-Kramers formula has been proved in [2,3,9], the case of diverging volume presents the typical features of sharp interface limits. We refer to

[32] for a heuristic discussion and to [25] for the analogous problem in the context of lattice models.

To observe the metastable behavior mentioned above, the functional $\mathcal{F}$ must possess at least two local minima. This condition is ensured by our choice of free boundary conditions, in contrast to Dobrushin boundary conditions—that is, non-homogeneous Dirichlet boundary conditions fixed at ± 1—as considered, for instance, in [5,30,40]. Under Dobrushin boundary conditions, the minimizer of $\mathcal{F}$ is unique in finite volume. However, in the infinite volume limit, $\mathcal{F}$ admits a manifold M of minimizers of the form $M = \{\bar{m}(\cdot - z) \colon z \in \mathbb{R}\}$, where $\bar{m}$ is the heteroclinic orbit of the Euler-Lagrange equations connecting the equilibria ± 1 (for example, $\bar{m}(x) = \tanh x$ in the paradigmatic case $W(\phi) = \frac{1}{2}(\phi^2 - 1)^2$).

Accordingly, the measure $\mu_{\beta,\ell}$ concentrates around the unique minimizer as $\beta \to \infty$ with fixed ℓ, whereas different asymptotic behaviors emerge when ℓ diverges with β. In [40], it is shown that if $\ell \sim \beta^{\gamma}$ with $\gamma \in (0, \frac{2}{3})$, the measure concentrates on the manifold of minimizers M of $\mathcal{F}$, and as demonstrated in [30], the distribution of the interface is uniform.

The situation changes at much smaller scales, as analyzed in [5], where it is proven that there exists a critical threshold γ_W (with $\gamma_W = \frac{1}{4}$ in the paradigmatic case) such that when $\ell = \gamma_W \log \beta$ the measure has a non-trivial limit that retains memory of the boundary conditions. In this regime, the limiting measure is not translation invariant and describes a localized interface. Conversely, as shown in [30], when $\log \beta \ll \ell \ll \bar{\ell}$, the probability of observing a transition layer from -1 to $+1$ within a given subinterval becomes independent of the location of the latter.

From a dynamical perspective, the presence or absence (depending on the choice of ℓ) of boundary effects is similarly observed in the stochastic Allen–Cahn equation when analyzing the random motion of the interface separating the pure phases. In this context, this corresponds to the emergence or lack of a nonlinear drift reflecting a repulsive interaction with the boundary; see, for instance, [4,8,18].

Finally, we comment on the symmetry assumption of the double-well potential W. Even when the two wells are of equal depth, introducing a slight asymmetry can significantly alter the system's behavior as the system size ℓ tends to infinity. In this regime, the ϕ_1^4 measure may concentrate around one of the two wells. This phenomenon is closely related to the behavior of the ground state of the corresponding quantum anharmonic oscillator in the semiclassical limit, see [37]. From a dynamical standpoint, such asymmetry manifests in the stochastic Allen-Cahn equation as a drift in the random motion of the interface separating the pure phases, as discussed in [7]. On the other hand, it remains an open and intriguing question whether the results presented in this paper persist when a small asymmetry—vanishing as $\beta \to \infty$—is introduced into the potential.

2. Notation and Results

Given strictly positive sequences $a_n, b_n, n \in \mathbb{N}$, we use the standard asymptotic notation $a_n \sim b_n$ when $a_n = b_n(1 + r_n)$, with $\lim_n r_n = 0$. Given two sequences $a_n(x), b_n(x)$ depending on the parameter $x \in \mathcal{X}$, we say $a_n(\cdot) \sim b_n(\cdot)$ uniformly in

$\mathcal{X}$ when $a_n(x) = b_n(x)[1 + r_n(x)]$, with $\lim_n \sup_{x\in\mathcal{X}} |r_n(x)| = 0$. For $p \in [1, \infty)$, $\delta > 0$, and $\mathcal{A} \subset L^p((0, 1))$, the δ-neighborhood of $\mathcal{A}$ is denoted by $\mathcal{O}^p_\delta(\mathcal{A})$.

For the sake of concreteness, we here stick to the case of the paradigmatic quartic choice of the double well potential, namely,

$$W(\phi) := \frac{1}{2}(\phi^2 - 1)^2. \tag{2.1}$$

For the sake of readability, we however write the dependence on W of the relevant constants in the general case of symmetric double well potential. As the arguments in the proofs are based on the analysis of stochastic processes, we denote by t the space variable in $[0, \ell]$ and by s the rescaled variable in $[0, 1]$. Correspondingly, we denote by X the order parameter.

For $\ell > 0$, we define the map $\iota_\ell \colon (0, \ell) \to (0, 1)$ as the dilation $t \mapsto t/\ell$ and use the same notation for its lift to functions, that is, $(\iota_\ell X)(s) = X(\ell s)$, $s \in (0, 1)$. We then define the rescaled free energy functional

$$F_\ell(X) := \mathcal{F}_\ell\big(\iota_\ell^{-1}(X)\big) = \int_0^1 \mathrm{d}s \left[\frac{1}{2\ell}\dot{X}_s^2 + \ell\, W(X_s)\right], \tag{2.2}$$

that we regard as a lower semicontinuous functional on $L^p((0, 1))$, $p \in [1, \infty)$, understanding that $F_\ell(X) = +\infty$ if X does not belong to $H^1((0, 1))$. By the celebrated Modica–Mortola result [28], in the limit $\ell \to \infty$ the Γ-limit of the sequence $(F_\ell)_{\ell>0}$ is the functional $F \colon L^p((0, 1)) \to [0, +\infty]$ defined by

$$F(X) = \begin{cases} C_\mathrm{W}|S(X)| & \text{if } X \in BV((0, 1); \{-1, 1\}) \\ +\infty & \text{otherwise,} \end{cases} \tag{2.3}$$

where, for $X \in BV((0, 1))$, the space of real-valued functions of bounded variation on $(0, 1)$, we have denoted by $S(X)$ the *jump set* of X and by $|S(X)|$ its cardinality, see, for example, [1]. Finally, C_W is the surface tension in the van der Waals theory, given by

$$C_\mathrm{W} := \int_{-1}^1 \mathrm{d}x\, \sqrt{2W(x)}. \tag{2.4}$$

In particular, $C_\mathrm{W} = 4/3$ for the choice (2.1) of the double well potential.

We next define precisely the ϕ^4_1 measure on the interval $(0, \ell)$ with inverse temperature β and free boundary conditions, informally introduced in (1.1). Let D^2_N be the realization of the second derivative in $L^2((0, \ell))$ with Neumann boundary condition. For $\beta \in (0, \infty)$ consider the trace class operator $C_{\beta,\ell} = \beta^{-1}(-D^2_\mathrm{N} + 1)^{-1}$ and denote by $\mu^0_{\beta,\ell}$ the Gaussian measure on $C\big([0, \ell]\big)$ with mean zero and covariance $C_{\beta,\ell}$.

Setting $W_0(x) := \frac{1}{2}x^2$ and $\hat{W} := W - W_0$, the ϕ^4_1 measure $\mu_{\beta,\ell}$ is then defined in terms of the Radon–Nykodym derivative with respect to $\mu^0_{\beta,\ell}$ by

$$\mathrm{d}\mu_{\beta,\ell} = \frac{1}{Z_{\beta,\ell}} \exp\Big\{ -\beta \int_0^\ell \mathrm{d}t\, \hat{W}(\cdot)\Big\}\, \mathrm{d}\mu^0_{\beta,\ell}, \qquad Z_{\beta,\ell} = \mu^0_{\beta,\ell}\Big(\mathrm{e}^{-\beta\int_0^\ell \mathrm{d}t\, \hat{W}(\cdot)}\Big). \tag{2.5}$$

Our present purpose is to investigate the behavior of the probability $\mu_{\beta,\ell}$ in the joint limit $\beta, \ell \to \infty$. Let $\bar{\ell}$ be the typical length of the transition between the pure phases. As we show in Theorem 4.6 below, it is given by

$$\bar{\ell} = \bar{\ell}_\beta := \frac{2}{A_{\mathrm{W}}\sqrt{\beta}} \exp\{\beta C_{\mathrm{W}}\}, \tag{2.6}$$

where

$$A_{\mathrm{W}} := \frac{2\sqrt{2}}{\sqrt{\pi}} \sqrt{W(0)\sqrt{W''(1)}} \exp\left\{ - \int_0^1 \mathrm{d}x \, \frac{W'(x) - \sqrt{2W''(1)W(x)}}{2W(x)} \right\}. \tag{2.7}$$

Since W has a quadratic minimum for $x = 1$, the function inside the integral in (2.7) is regular. For the standard choice (2.1), $A_{\mathrm{W}} = 8\sqrt{2/\pi}$. We refer, for example, to [21, Eq. (4.5.22)] for an equivalent definition of the constant A_W.

As we soon discuss, the asymptotic behavior of $\mu_{\beta,\ell}$ is quite different in the regimes $\ell \ll \bar{\ell}$ and $\ell \gg \bar{\ell}$.

Large Deviations in the Regime $\boldsymbol{\beta \ll \ell \ll \bar{\ell}}$

In this situation, by the symmetry of the ϕ^4_1 measure with free boundary conditions with respect to $X \mapsto -X$, the probability $\mu_{\beta,\ell}$ concentrates on the pure phases ± 1. Our first result establishes the corresponding large deviation principle. To this end, fix a sequence $(\ell_\beta)_{\beta>0}$ satisfying

$$\lim_{\beta\to\infty} \beta^{-1}\ell_\beta = \infty \quad \text{and} \quad \lim_{\beta\to\infty} \beta^{-1}\log \ell_\beta = \alpha \in [0, C_{\mathrm{W}}). \tag{2.8}$$

The first condition above is assumed for technical reasons and in fact the large deviation statement should hold whenever $\ell_\beta \to \infty$.

Recalling that the map $\imath_\ell$ is defined above (2.2), we shorthand $\imath_\beta = \imath_{\ell_\beta}$ and introduce the rescaled ϕ^4_1 measure as

$$\mu_\beta := \mu_{\beta,\ell_\beta} \circ \imath_\beta^{-1}, \tag{2.9}$$

which we regard as a one-parameter family of probability measures on $L^p((0,1))$, $p \in [1,\infty)$. For ℓ fixed, by the representation (2.5), the Laplace–Varadhan lemma implies that the family $\left(\mu_{\beta,\ell}\right)_{\beta>0}$ satisfies a large deviations principle with speed β and rate function $\mathcal{F}_\ell$. In view of the variational convergence of the sequence of functionals (F_ℓ), we then deduce that μ_β satisfies a large deviation principle with speed β and rate function F as in (2.3) when the limit $\ell \to \infty$ is taken after the limit $\beta \to \infty$. We here show that this statement holds whenever ℓ_β grows sub-exponentially in β. On the other hand, as first discussed in [30], if ℓ_β grows exponentially in β, namely $\alpha > 0$ in (2.8), then an entropic effect does modify the rate function. More precisely, the modified rate function $I \colon L^p((0,1)) \to [0,+\infty]$ is given by

$$I(X) := \begin{cases} (C_{\mathrm{W}} - \alpha)|S(X)| & \text{if } X \in BV((0,1);\{-1,1\}) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.10}$$

In view of (2.8) and the entropy in the location of the interfaces, this rate function reflects the heuristic that the probability of n interfaces is roughly $\ell_\beta^n \mathrm{e}^{-n\beta C_{\mathrm{W}}}$.

Theorem 2.1. *Assume* (2.8) *and fix* $p \in [1, \infty)$. *The family* $\left(\mu_\beta\right)_{\beta>0}$ *of probability measures on* $L^p((0, 1))$ *satisfies a large deviation principle with speed* β *and the good rate function* I *in* (2.10). *Namely,* I *has compact level sets and for each closed* $\mathcal{C} \subset L^p((0, 1))$ *and open* $\mathcal{O} \subset L^p((0, 1))$,

$$\varlimsup_{\beta\to\infty} \frac{1}{\beta} \log \mu_\beta(\mathcal{C}) \le - \inf_{X\in\mathcal{C}} I(X), \qquad \varliminf_{\beta\to\infty} \frac{1}{\beta} \log \mu_\beta(\mathcal{O}) \ge - \inf_{X\in\mathcal{O}} I(X).$$

The choice of the free boundary conditions for the ϕ^4_1 measure is not particularly relevant for the validity of the large deviation principle. The arguments used in proof of Theorem 2.1 cover directly the case in which $\mu_{\beta,\ell}$ is the restriction to the interval $[0, \ell]$ of the infinite volume ϕ^4_1 measure. It is also possible to deduce the large deviations for $\mu^{a,b}_{\beta,\ell}$, the ϕ^4_1 measure with the Dirichlet boundary conditions $X_0 = a$, $X_\ell = b$, $a, b \in \mathbb{R}$; in this case the rate function depends however on the signs of a and b. We do not pursue this issue here.

Sharp Asymptotics in the Regime $\boldsymbol{\beta^2 \ll \ell \ll \bar{\ell}}$

The next result describes the sharp asymptotics of the μ_β probability of neighborhoods of the family of profiles with n transitions between the pure phases. More precisely, we show that this probability behaves as if the number of transitions were a Poisson random variable with parameter $\ell/\bar{\ell}$. To this end, we fix a sequence $(\ell_\beta)_{\beta>0}$ satisfying

$$\lim_{\beta\to\infty} \beta^{-2}\ell_\beta = \infty \quad \text{and} \quad \lim_{\beta\to\infty} \frac{\ell_\beta}{\bar{\ell}_\beta} = 0. \tag{2.11}$$

The first requirement is technical and an arbitrary power law growth on ℓ_β should suffice. On the other hand, when ℓ_β grows only logarithmically in β other phenomena become relevant, we refer to the analysis in [5] for the case of Dobrushin boundary conditions.

We start by introducing the family of profiles with n transitions. For $n = 0$ we set $m^{\pm,0} = \pm 1$. For $n \in \mathbb{N}$ and $0 < s_1 < \cdots < s_n < 1$, let $m^{\pm,n}_{s_1,\ldots,s_n} : (0, 1) \to \{-1, 1\}$ be the profile with n jumps at times $s_1, \ldots, s_n$ defined by

$$m^{\pm,n}_{s_1,\ldots,s_n} = \pm(-1)^{k+1}, \text{ on } [s_{k-1}, s_k),\ k = 1, \ldots, n + 1, \tag{2.12}$$

where we understand that $s_0 = 0$ and $s_{n+1} = 1$. Let finally $\mathcal{M}_n$ be the manifold of the profiles in $BV((0, 1); \{-1, 1\})$ with n jumps, that is,

$$\mathcal{M}_n = \mathcal{M}^+_n \cup \mathcal{M}^-_n, \qquad \mathcal{M}^\pm_n = \bigcup_{0<s_1<\cdots<s_n<1} \{m^{\pm,n}_{s_1,\ldots,s_n}\}, \tag{2.13}$$

where we understand $\mathcal{M}_0 = \{m^{+,0}, m^{-,0}\}$. We finally recall that $\mathcal{O}^p_\rho(\mathcal{B})$ denotes the ρ-neighborhood of $\mathcal{B} \subset L^p((0, 1))$.

Theorem 2.2. *Assume* (2.11)*, fix* $p \in [1, \infty)$ *and* $n \geq 0$*. Let also* $(\rho_{k,\beta})_{\beta>0}$*,* $k = 1, 2$*, be sequences satisfying* $\lim_\beta(\sqrt{\beta} \wedge \beta^{1/p})\rho_{k,\beta} = \infty$*,* $k = 1, 2$*, and* $\lim_\beta \rho_{1,\beta} = 0$*. Then, as* $\beta \to \infty$*,*

$$\mu_\beta\big(\mathcal{O}^p_{\rho_{2,\beta}}(\mathcal{M}_n) \setminus \mathcal{O}^p_{\rho_{1,\beta}}(\mathcal{M}_{n-1})\big) \sim \frac{1}{n!}\left(\frac{\ell_\beta}{\bar{\ell}_\beta}\right)^n = \frac{\big(\ell_\beta A_\mathrm{W}\sqrt{\beta}\big)^n}{2^n n!}\mathrm{e}^{-n\beta C_\mathrm{W}}.$$

In the above statement, for $n = 0$, it is understood that $\mathcal{O}^p_{\rho_{1,\beta}}(\mathcal{M}_{-1}) = \emptyset$. The condition $\rho_{1,\beta} \to 0$ is required to allow enough entropy on the location of the interfaces to pick the prefactor ℓ^n. The condition $\rho_{k,\beta} \gg \beta^{-1/2} \vee \beta^{-1/p}$, $k = 1, 2$, is required to accommodate for the typical fluctuations of the probability μ_β in $L^p((0, 1))$, see Lemma 4.2 (ii).

As in the case of Theorem 2.1, the arguments used to deduce the sharp asymptotics cover directly the case in which $\mu_{\beta,\ell}$ is the restriction to the interval $[0, \ell]$ of the infinite volume ϕ^4_1 measure. In contrast, the analysis cannot be applied to the probability $\mu^{a,b}_{\beta,\ell}$ with Dirichlet boundary conditions, as the crucial ingredient provided by Theorem 4.5 cannot be applied.

The Regime $\ell \gg \bar{\ell}$

To describe the asymptotic behavior of the ϕ^4_1 measure in this regime, it will be convenient to regard the probability $\mu_{\beta,\ell}$ defined on the interval $[-\ell/2, \ell/2]$ rather than $[0, \ell]$. We further consider $\mu_{\beta,\ell}$ as a probability on $C(\mathbb{R})$ by understanding $X_t = X_{-\ell/2}$ for $t < -\ell/2$ and $X_t = X_{\ell/2}$ for $t > \ell/2$.

Fix a sequence $(\ell_\beta)_{\beta>0}$ such that

$$\lim_{\beta\to\infty} \frac{\ell_\beta - \bar{\ell}_\beta}{\beta} = +\infty. \tag{2.14}$$

We here consider the rescaled ϕ^4_1 measure defined by

$$\bar{\mu}_\beta := \mu_{\beta,\ell_\beta} \circ \iota^{-1}_{\bar{\ell}_\beta}.$$

For $p \in [1, \infty)$, we regard $(\bar{\mu}_\beta)_{\beta>0}$ as a sequence of probability measures on $L^p_\mathrm{loc}(\mathbb{R})$. As we next state, this sequence converges weakly to some not trivial probability measure. In order to describe the limiting probability, we consider the continuous-time Markov chain with state space $\{-1, 1\}$ and generator

$$\bar{L} = \begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix}.$$

We denote by $\bar{\mu}$ the law of the stationary process associated to $\bar{L}$, regarded as a probability measure on $L^p_\mathrm{loc}(\mathbb{R})$. Equivalently, given the family $\Sigma = (s_k)_{k\in\mathbb{Z}}$ satisfying $s_k < s_{k+1}$, $k \in \mathbb{Z}$, and $\lim_{k\to\pm\infty} s_k = \pm\infty$, let $m^\pm_\Sigma$ be the element of $L^p_\mathrm{loc}(\mathbb{R})$ defined by

$$m^\pm_\Sigma = \pm\sum_{k\in\mathbb{Z}}(\pm 1)^k \mathbf{1}_{[s_k,s_{k+1})}.$$

Denoting by $\bar{\mu}^{\pm}$ the probability on $L^p_{\text{loc}}(\mathbb{R})$ concentrated on $m^{\pm}_{\Sigma}$ when Σ is sampled according to a Poisson point process on $\mathbb{R}$ with intensity one, then $\bar{\mu} = \frac{1}{2}\bar{\mu}^{+} + \frac{1}{2}\bar{\mu}^{-}$.

Theorem 2.3. *Assume* (2.14) *and fix* $p \in [1, \infty)$. *As* $\beta \to \infty$ *the sequence* $(\bar{\mu}_\beta)_{\beta>0}$ *of probability measures on* $L^p_{\text{loc}}(\mathbb{R})$ *converges weakly to* $\bar{\mu}$.

In contrast to Theorem 2.2, the choice of free boundary condition for $\mu_{\beta,\ell}$ is not relevant for Theorem 2.3. The same limiting probability is obtained for any, ℓ-independent, choice of the boundary conditions or directly for the infinite volume ϕ^4_1 measure. We also note that in the critical case $\ell_\beta = \bar{\ell}_\beta$, Theorem 2.3 still holds with the following modifications. The probability $\bar{\mu}_\beta$ is regarded as a probability on $L^p((-1/2, 1/2))$ and $\bar{\mu}$ is the marginal on the interval $(-1/2, 1/2)$ of the probability described above.

As can be easily checked, the statement of Theorem 2.3 does not hold in the Skorohod topology. The convergence to the limiting measure $\bar{\mu}$ is a generic feature of bistable reversible diffusions in the small noise limit [27,31,33]. The choice of the L^p topology on the space of paths appears however novel. In the context of the one-dimensional Kac–Ising model, the statement analogous to Theorem 2.3 has been proven in [10], see also [34] for the one-dimensional nearest neighbors Ising model.

Outline of Proofs and Organization of the Paper

As well known [38], the infinite volume ϕ^4_1 measure is the law of the stationary process associated to the one-dimensional reversible diffusion described by the stochastic differential equation

$$\mathrm{d}X_t = -\frac{1}{\beta}\big(\log \psi_{0,\beta}\big)'(X_t)\mathrm{d}t + \sqrt{\beta^{-1}}\mathrm{d}w_t, \tag{2.15}$$

where w is a standard Brownian motion on $\mathbb{R}$ and $\psi_{0,\beta}$ is the ground state of the quantum anharmonic oscillator with Hamiltonian $H_\beta = -\frac{1}{2\beta}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta W$. As we detail below, also the finite volume ϕ^4_1 measure with free boundary conditions can be represented in terms of the solutions to (2.15).

The present analysis is based upon sharp estimates for the metastable behavior of the solution to (2.15) in the small noise limit, which are the content of Section 4. As first exploited in [24], these estimates are related to the sharp semiclassical analysis of the Schrödinger operator H_β, here developed in Section 3, by applying the ground state transformation. In particular, as $\beta \to \infty$, the typical length $\bar{\ell}_\beta$ can be characterized as twice the inverse of the spectral gap of H_β. In contrast to the semiclassical analysis in the standard setting of reversible diffusions, see, for example, [14,15] and references therein, here the potential in the Schrödinger operator does not depend on β, while the drift $-\beta^{-1}(\log \psi_{0,\beta})'$ depends on the semiclassical parameter and exhibits a discontinuity at the saddle point in the limit $\beta \to \infty$.

The proof of Theorem 2.1, given in Section 5, will be achieved by combining the large deviation asymptotics of the solution to (2.15) over a fixed time window with the behavior of the discrete time Markov chain that is defined by sampling X at the transition times between the pure phases. In this respect, some of the arguments are similar to those used in [17, Chapter 4].

Theorems 2.2 and 2.3 are proved in Section 6 and 7, respectively. For both of them the sharp asymptotics, that includes the prefactor of the transition time for the diffusion (2.15), is relevant. More precisely, Theorem 2.3 essentially follows from the convergence in law of the suitably rescaled transition time to an exponential random variable. For Theorem 2.2, it will be instead relevant to deduce the asymptotic probability of having n transitions between the pure phases in the time interval $[0, \ell]$. As stated above, the latter point will be derived from the sharp semiclassical analysis of the quantum anharmonic oscillator with potential W given in Section 3. This includes, as a novel point, a uniform control on the difference between the first two eigenfunctions in terms of the spectral gap, see Theorem 3.6.

3. Sharp Semiclassical Analysis of the Anharmonic Oscillator

Within this section, we denote by $L^p = L^p(\mathbb{R})$, $p \in [1, \infty]$ and $H^1 = H^1(\mathbb{R})$ the standard Lebesgue and Sobolev spaces on $\mathbb{R}$. The inner product in L^2 is denoted by $\langle \cdot, \cdot \rangle$.

Let $H = H_\beta$ be the selfadjoint Schrödinger operator on L^2 which on its core $C_c^2(\mathbb{R})$ is given by

$$H_\beta = -\frac{1}{2\beta}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta W. \tag{3.1}$$

Since W has compact level sets, H_β has purely discrete spectrum. We denote by $E_k = E_{k,\beta}$, $k \in \mathbb{Z}_+$, the corresponding eigenvalues enumerated in increasing order and by $\psi_k = \psi_{k,\beta}$, $k \in \mathbb{Z}_+$, a corresponding orthonormal basis of eigenfunctions. By the symmetry of W, $\psi_k(x) = (-1)^k \psi_k(-x)$. In the sequel, we choose the ground state ψ_0 to be positive and ψ_1 to be positive on $\mathbb{R}_+$.

We set

$$U_\beta = -\frac{1}{\beta}\log \psi_{0,\beta} \tag{3.2}$$

and observe that, as follows from a direct computation, it satisfies the Riccati equation

$$\frac{1}{2}(U_\beta')^2 - \frac{1}{2\beta}U_\beta'' = W - \frac{1}{\beta}E_{0,\beta}. \tag{3.3}$$

We further define

$$U(x) = \min\left\{\left|\int_{-1}^{x}\mathrm{d}y\,\sqrt{2W(y)}\right|, \left|\int_{1}^{x}\mathrm{d}y\,\sqrt{2W(y)}\right|\right\}, \tag{3.4}$$

which represents the Agmon distance of x from the two wells -1 and 1. In particular, recalling (2.4), $C_\mathrm{W} = 2U(0)$. For the special choice of the double-well potential (2.1), $U(x) = \frac{1}{3}|1 - |x|^2|(2 + |x|)$. As we shall see in Proposition 3.1 below, $U_\beta \to U$ as $\beta \to \infty$. Note that U_β, being solution of the elliptic equation (3.3) is smooth, while U exhibits a corner singularity (at $x = 0$). This is not surprising as it is the viscosity solution to the eikonal equation $|U'| = \sqrt{2W}$ with boundary condition $U(-1) = U(1) = 0$.

Let $\bar{E}_k := (k + \frac{1}{2})\sqrt{W''(\pm 1)} = 2k + 1$, $k \in \mathbb{Z}_+$, be the eigenvalues of the harmonic oscillators centered in ±1 defined by the Hamiltonians $-\frac{1}{2\beta}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\beta}{2}W''(\pm 1)(x \mp 1)^2$ and denote by

$$g_\pm(x) = \Big(\beta\pi^{-1}\sqrt{W''(\pm 1)}\Big)^{1/4} \exp\Big\{ -\frac{1}{2}\beta\sqrt{W''(\pm 1)}\,\big(x \mp 1\big)^2\Big\}$$

the corresponding ground states. Finally, we introduce the functions

$$g_0 = \frac{1}{Z_{0,\beta}}\frac{1}{\sqrt{2}}(g_+ + g_-), \qquad g_1 = \frac{1}{Z_{1,\beta}}\frac{1}{\sqrt{2}}(g_+ - g_-),$$

where $Z_{k,\beta}$ is chosen so that $\|g_k\|_{L^2} = 1$, $k = 0, 1$. Note that g_0 and g_1 are respectively even and odd. Furthermore, $Z_{k,\beta} \sim 1$, $k = 0, 1$.

In the next statement we summarize the properties of H_β that are relevant for the present analysis. Recall that the constant A_{W} has been introduced in (2.7).

Proposition 3.1.

(i) For each $k \in \mathbb{Z}_+$,

$$\lim_{\beta\to\infty} E_{k,\beta} = \begin{cases} \bar{E}_{k/2} = k + 1 & \text{if } k \text{ is even,} \\ \bar{E}_{(k-1)/2} = k & \text{if } k \text{ is odd.} \end{cases}$$

(ii) As $\beta \to \infty$,

$$E_{1,\beta} - E_{0,\beta} \sim A_{\mathrm{W}}\sqrt{\beta}\,\mathrm{e}^{-\beta C_{\mathrm{W}}}.$$

(iii) Uniformly on compact sets, $U_\beta \to U$ as $\beta \to \infty$.

(iv) There exist $R, \gamma \in (0, \infty)$ such that, for any $\beta \geq 1$ and $|x| \geq R$,

$$|\psi_k(x)| \leq \mathrm{e}^{-\beta\gamma|x|}, \qquad k = 0, 1.$$

(v) There exists $C \in (0, \infty)$ such that, for any $\beta \geq 1$,

$$\|\psi_k - g_k\|_{L^2}^2 \leq C\beta^{-1}, \qquad \|\psi_k - g_k\|_{L^\infty}^2 \leq C, \qquad k = 0, 1.$$

We refer to [11,22,35] for the proof of item (i), to [19–21] for (ii), to [36] for (iii) and (iv). For completeness, the proof of (v) will be detailed after a preliminary lemma.

For $R > 0$ we denote by $H^R = H^R_\beta$ the selfadjoint Schrödinger operator $-\frac{1}{2\beta}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta W$ on $L^2((-R, R))$ with Dirichlet boundary conditions at the endpoints. We understand $H^R = H$ when $R = \infty$. We denote by $E^R_0 = E^R_{0,\beta}$ the bottom of the spectrum of H^R and, for $c \geq E^R_0$, by $\mathbf{1}_{[c,\infty)}(H^R)$ the spectral projector associated to H^R on the interval $[c, \infty)$.

Lemma 3.2. *Fix $R \in (0, \infty]$.*

(i) Let $0 \leq E \leq E_0^R < c$ *and* f *be in the form domain of* H^R*. Then, for any* $\beta > 0$*,*

$$\|\mathbf{1}_{[c,\infty)}(H^R)f\|^2_{L^2((-R,R))} \leq \frac{1}{c-E}\|(H^R-E)^{\frac{1}{2}}f\|^2_{L^2((-R,R))}.$$

(ii) Let $0 \leq E < c$ *and* f *be in the domain of* H^R*. Then, for any* $\beta > 0$*,*

$$\begin{aligned}&\|\mathbf{1}_{[c,\infty)}(H^R)f\|^2_{L^\infty((-R,R))}\\&\quad\leq \frac{1}{c-E}\left(2\beta+\frac{2\beta E+1}{c-E}\right)\|(H^R-E)f\|^2_{L^2((-R,R))}.\end{aligned}$$

Proof. The first statement follows directly from the spectral theorem and the Chebyshev inequality. To prove (ii), let $g = \mathbf{1}_{[c,\infty)}(H^R)f$ and denote by $(P_\lambda)_{\lambda\geq 0}$ the projection-valued measure associated to H^R. Since $W \geq 0$,

$$\begin{aligned}&\|g\|^2_{H_0^1((-R,R))}\\&\leq 2\beta\int_{-R}^{R}\mathrm{d}x\left[\frac{1}{2\beta}(g')^2+(\beta W-E)g^2\right]+(2\beta E+1)\|g\|^2_{L^2((-R,R))}\\&=2\beta\int_c^\infty(\lambda-E)\mathrm{d}\langle g,P_\lambda g\rangle_{L^2((-R,R))}+(2\beta E+1)\int_c^\infty \mathrm{d}\langle g,P_\lambda g\rangle_{L^2((-R,R))}\\&\leq\frac{2\beta}{c-E}\|(H^R-E)g\|^2_{L^2((-R,R))}+\frac{2\beta E+1}{(c-E)^2}\|(H^R-E)g\|^2_{L^2((-R,R))}\\&\leq\frac{2\beta}{c-E}\|(H^R-E)f\|^2_{L^2((-R,R))}+\frac{2\beta E+1}{(c-E)^2}\|(H^R-E)f\|^2_{L^2((-R,R))}.\end{aligned}$$

The conclusion follows by the Sobolev embedding. □

Proof of Proposition 3.1 (v). Fix $k = 1, 2$ and set $R_k = g_k - \langle g_k, \psi_k\rangle\psi_k$ Observe that by symmetry R_k is orthogonal to both ψ_0 and ψ_1. Further, since $\langle g_k, \psi_k\rangle \geq 0$,

$$|\langle g_k,\psi_k\rangle - 1| = \left|\frac{\langle g_k,\psi_k\rangle^2-1}{\langle g_k,\psi_k\rangle+1}\right| \leq \|R_k\|^2_{L^2}.$$

Hence

$$\|g_k-\psi_k\|_{L^2} \leq \|R_k\|_{L^2}+\|R_k\|^2_{L^2}$$

and

$$\|g_k-\psi_k\|_{L^\infty} \leq \|R_k\|^2_{L^2}\|\psi_k\|_{L^\infty}+\|R_k\|_{L^\infty}. \tag{3.5}$$

To prove the first bound we apply Lemma 3.2 (i) with $R = \infty$, $E = \bar{E}_0 = 1$, $f = g_k$ and $c \in (\bar{E}_0, \bar{E}_1)$ so that $R_k = \mathbf{1}_{[c,\infty)}(H)g_k$. We deduce

$$\|R_k\|^2_{L^2} \leq \frac{1}{c-\bar{E}_0}\|(H-\bar{E}_0)^{\frac{1}{2}}g_k\|^2_{L^2}.$$

By a direct computation,

$$\|(H-\bar{E}_0)^{\frac{1}{2}}g_\pm\|^2_{L^2} = \langle (H-\bar{E}_0)g_\pm, g_\pm\rangle_{L^2} = \beta\int \mathrm{d}x\,\left(\tfrac{1}{2}x^4 \pm 2x^3\right)\sqrt{\frac{2\beta}{\pi}}\mathrm{e}^{-2\beta x^2}.$$

It follows that there is a constant C independent of β such that $\|(H-\bar{E}_0)^{\frac{1}{2}}g_k\|_{L^2}^2 \leq C/\beta$, which concludes the proof of the first bound.

To prove the second bound we apply Lemma 3.2 (ii) again with $R=\infty$, $E=\bar{E}_0=1$, $f=g_k$ and $c\in(\bar{E}_0,\bar{E}_1)$. We deduce

$$\|R_k\|_{L^\infty}^2 \leq \frac{1}{c-\bar{E}_0}\left(2\beta+\frac{2\beta\bar{E}_0+1}{c-\bar{E}_0}\right)\|(H-\bar{E}_0)g_k\|_{L^2}^2.$$

By a direct computation,

$$\|(H-\bar{E}_0)g_\pm\|_{L^2}^2=\beta^2\int \mathrm{d}x\,\left(\tfrac{1}{2}x^4\pm 2x^3\right)^2\sqrt{\frac{2\beta}{\pi}}\mathrm{e}^{-2\beta x^2}.$$

It follows that there is a constant C independent of β such that $\|(H-\bar{E}_0)g_k\|_{L^2}^2 \leq C/\beta$. Further, by the Sobolev embedding and Proposition 3.1 (i), $\|\psi_k\|_{L^\infty}\leq C\sqrt{\beta}$. We conclude recalling (3.5). □

The estimates stated below are readily deduced from Proposition 3.1 (iv) and the first bound in Proposition 3.1 (v).

Corollary 3.3. *Let $I_\pm$ be neighborhoods of ± 1 such that ∓ 1 does not belong to the closure of $I_\pm$. Then, for $k=0,1$, as $\beta\to\infty$,*

$$\int_{I_+}\mathrm{d}x\,\psi_{k,\beta}(x)\sim\left(\frac{\pi}{2\beta}\right)^{1/4},\qquad \int_{I_-}\mathrm{d}x\,\psi_{k,\beta}(x)\sim(-1)^k\left(\frac{\pi}{2\beta}\right)^{1/4},\tag{3.6}$$

$$\pm\int_{I_\pm}\mathrm{d}x\,\psi_{0,\beta}(x)\psi_{k,\beta}(x)\sim\frac{1}{2}.\tag{3.7}$$

Moreover,

$$\int_{\mathbb{R}}\mathrm{d}x\,\psi_{0,\beta}(x)\sim 2\left(\frac{\pi}{2\beta}\right)^{1/4},\qquad \int_{\mathbb{R}}\mathrm{d}x\,\psi_{1,\beta}(x)=0.\tag{3.8}$$

The semigroup $\left(\mathrm{e}^{-t(H-E_0)}\right)_{t\geq 0}$ admits a smooth kernel $g_t(x,y)=g_t^\beta(x,y)$, $t>0$, $x,y\in\mathbb{R}$, that can be represented as

$$g_t(x,y)=\sum_{k=0}^{\infty}\mathrm{e}^{-t(E_k-E_0)}\psi_k(x)\psi_k(y).\tag{3.9}$$

Given ℓ and $T\in[0,\ell/2)$, let $\wp^T=\wp^T_{\beta,\ell}$ be the probability measure on $\mathbb{R}^2$ with density $\varrho^T=\varrho^T_{\beta,\ell}$ given by

$$\varrho^T(x,y)=\frac{1}{Z_\ell}\int\mathrm{d}a\,\mathrm{d}b\,g_T(a,x)g_{\ell-2T}(x,y)g_T(y,b),\quad Z_\ell=\int\mathrm{d}a\,\mathrm{d}b\,g_\ell(a,b).\tag{3.10}$$

When $T=0$ we write $\wp$ for $\wp^T$ and ϱ for ϱ^T so that $\varrho(x,y)=Z_\ell^{-1}g_\ell(x,y)$. As discussed in [38], $\wp^T$ is the law of $(X_T,X_{\ell-T})$ when X is sampled according to

$\mu_{\beta,\ell}$. As we next prove, when $T \gg \beta$ the probability $\wp^T$ is well approximated by the probability $\bar{\wp}^T$ whose density $\bar{\varrho}^T = \bar{\varrho}^T_{\beta,\ell}$ is given by

$$\bar{\varrho}^T(x, y) = \psi_0(x) g_{\ell-2T}(x, y) \psi_0(y). \tag{3.11}$$

Note that the probability $\bar{\wp}^T$ is the marginal at the times T and $\ell - T$ of the law of the infinite volume ϕ^4_1 measure.

Let also $\pi^T = \pi^T_\beta$ be the probability on $\mathbb{R}$ with density $q^T = q^T_\beta$ given by

$$q^T(x) = \frac{1}{\psi_0(0)} g_T(0, x) \psi_0(x), \tag{3.12}$$

which is the marginal at time T of the ϕ^4_1 measure on the half-line $[0, \infty)$ with boundary condition $X_0 = 0$. As we next prove, when $T \gg \beta$ the probability π^T is well approximated by the probability $\pi = \pi_\beta$ with density ψ_0^2, that is the single time marginal of the infinite volume ϕ^4_1 measure.

In the next statement we denote by $\|\cdot\|_{\mathrm{TV}}$ the total variation norm on the space of finite signed measures.

Proposition 3.4. *(i) There exists $\ell_0 > 0$ such that if $\underline{\lim}_\beta \ell_\beta \geq \ell_0$ then the sequence of probability measures $(\wp_{\beta,\ell_\beta})_{\beta>0}$ is exponentially tight as $\beta \to \infty$. Namely, there exists a sequence of compacts $K_R \subset\subset \mathbb{R}^2$ such that*

$$\lim_{R\to\infty} \overline{\lim_{\beta\to\infty}} \frac{1}{\beta} \log \wp_{\beta,\ell_\beta}(K_R^{\mathrm{c}}) = -\infty.$$

(ii) Assume that T_β satisfies $\lim_\beta \beta^{-1} T_\beta = \infty$ and ℓ_β be such that $\ell_\beta > 2T_\beta + 1$. Then

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \|\wp^{T_\beta}_{\beta,\ell_\beta} - \bar{\wp}^{T_\beta}_{\beta,\ell_\beta}\|_{\mathrm{TV}} = -\infty.$$

(iii) Assume that T_β satisfies $\lim_\beta \beta^{-1} T_\beta = \infty$. Then

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \|\pi^{T_\beta}_\beta - \pi_\beta\|_{\mathrm{TV}} = -\infty.$$

We remark that the exponential tightness in (i) actually holds for any $\ell_0 > 0$ but we refrain from proving this stronger statement as it is not needed in the present analysis. We also emphasize that the super-exponential bounds (ii) and (iii) depend on the invariance of the boundary conditions with respect to the map $X \mapsto -X$.

We present the following lemma based on the Golden–Thompson inequality [38, Theorem 9.2]:

Lemma 3.5. *There exists a constant C such that, for any $T \geq 1$ and β large enough,*

$$\sum_{k=2}^{\infty} \mathrm{e}^{-T(E_k - E_0)} \left(1 + \|\psi_k\|^2_{L^1}\right) \leq C \mathrm{e}^{-T}.$$

Proof. By the Golden–Thompson inequality, for each $\lambda > 0$,

$$\sum_{k=0}^{\infty} \mathrm{e}^{-\lambda E_k} \leq \frac{\beta}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}x\, \mathrm{d}p\, \mathrm{e}^{-\lambda\beta(\frac{1}{2}p^2 + W(x))} \leq C, \tag{3.13}$$

for a suitable constant $C = C(\lambda)$. On the other hand, for a suitable constant C, whose numerical value may change from line to line,

$$\begin{aligned}
\sum_{k=2}^{\infty} \mathrm{e}^{-T(E_k - E_0)} \|\psi_k\|_{L^1}^2 &= \sum_{k=2}^{\infty} \mathrm{e}^{-T(E_k - E_0)} \left(\int_{\mathbb{R}} \mathrm{d}x\, \frac{\sqrt{1 + W(x)}}{\sqrt{1 + W(x)}} |\psi_k(x)| \right)^2 \\
&\leq C \sum_{k=2}^{\infty} \mathrm{e}^{-T(E_k - E_0)} \int_{\mathbb{R}} \mathrm{d}x\, [1 + W(x)]\, |\psi_k(x)|^2 \\
&\leq C \sum_{k=2}^{\infty} \mathrm{e}^{-T(E_k - E_0)} \big(1 + \beta^{-1} E_k\big) \leq C \sum_{k=2}^{\infty} \mathrm{e}^{-(T - \beta^{-1}) E_k + T E_0} \\
&\leq C \mathrm{e}^{E_2} \mathrm{e}^{-T(E_2 - E_0)} \sum_{k=2}^{\infty} \mathrm{e}^{-(1 - \beta^{-1}) E_k}.
\end{aligned}$$

The statement follows then from (3.13) and Proposition 3.1 (i). □

Proof of Proposition 3.4. Proof of (i). By (3.9) and (3.8),

$$\varliminf_{\beta \to \infty} \frac{1}{\beta} \log Z_{\ell_\beta} \geq \varliminf_{\beta \to \infty} \frac{1}{\beta} \log \|\psi_0\|_{L^1}^2 = 0.$$

It is therefore enough to show that

$$\lim_{R \to \infty} \varlimsup_{\beta \to \infty} \frac{1}{\beta} \log \int_{K_R^c} \mathrm{d}x\, \mathrm{d}y\, g_{\ell_\beta}(x, y) = -\infty. \tag{3.14}$$

We claim that there exists a constant C such that, for every Borel subset $A \subset \mathbb{R}^2$ and $\beta \geq 1$,

$$\int_A \mathrm{d}x\, \mathrm{d}y\, g_{\ell_\beta}(x, y) \leq C \left(\int_A \mathrm{d}x\, \mathrm{d}y\, \big[\psi_0(x)\psi_0(y)\big]^{2/3} \right)^{3/4}. \tag{3.15}$$

This claim, together with Proposition 3.1 (iv) yields (3.14). In order to prove (3.15) we use that the semigroup $\big(\mathrm{e}^{-tH_\beta}\big)_{t>0}$ is *intrinsically hypercontractive*, uniformly in β. More precisely, by [12, Theorem 6.5], there exist $T, C \in (0, \infty)$ such that, for any $\beta \geq 1$,

$$\|\mathrm{e}^{TL_\beta} f\|_{L^4(\pi_\beta)} \leq C \|f\|_{L^2(\pi_\beta)},$$

where $L_\beta = -\frac{1}{\psi_0}(H_\beta - E_0)\psi_0$. In particular,

$$\left\| \frac{\psi_k}{\sqrt{\psi_0}} \right\|_{L^4} = \left\| \frac{\psi_k}{\psi_0} \right\|_{L^4(\pi_\beta)} \leq C \mathrm{e}^{T(E_k - E_0)}.$$

This last estimate, together with Hölder inequality, yields

$$\begin{aligned}\int_A \mathrm{d}x\,\mathrm{d}y\, g_{\ell_\beta}(x,y) &\leq \textstyle\sum_{k=0}^{\infty} \mathrm{e}^{-\ell_\beta(E_k-E_0)} \int_A \mathrm{d}x\,\mathrm{d}y\, \frac{|\psi_k(x)|}{\sqrt{\psi_0(x)}}\frac{|\psi_k(y)|}{\sqrt{\psi_0(y)}}\sqrt{\psi_0(x)}\sqrt{\psi_0(y)} \\ &\leq C^2 \textstyle\sum_{k=0}^{\infty} \mathrm{e}^{-(\ell_\beta-2T)(E_k-E_0)} \left(\int_A \mathrm{d}x\,\mathrm{d}y \left[\psi_0(x)\psi_0(y)\right]^{2/3}\right)^{3/4}.\end{aligned}$$

Choosing $\ell_0 > 2T$, the conclusion follows from the Golden–Thompson inequality (3.13) and Proposition 3.1 (i).

For the proof of (ii) and (iii) we first observe that, in view of (3.9), (3.8), and Lemma 3.5, the normalization in (3.10) satisfies

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \left| Z_{\ell_\beta} - \|\psi_0\|_{L^1}^2 \right| = -\infty \tag{3.16}$$

whenever $\lim_\beta \beta^{-1}\ell_\beta = \infty$.

Proof of (ii). Set

$$\mathcal{E}_{T_\beta}(x,y) = \int_{\mathbb{R}^2} \mathrm{d}a\,\mathrm{d}b\, g_{T_\beta}(a,x) g_{T_\beta}(y,b) - \|\psi_0\|_{L^1}^2\, \psi_0(x)\psi_0(y), \tag{3.17}$$

so that

$$Z_{\ell_\beta} \varrho^{T_\beta}(x,y) = \|\psi_0\|_{L^1}^2 \bar{\varrho}^{T_\beta}(x,y) + \mathcal{E}_{T_\beta}(x,y) g_{\ell_\beta-2T_\beta}(x,y).$$

By (3.9) and (3.8) we have

$$\begin{aligned}|\mathcal{E}_{T_\beta}(x,y)| \leq C\beta^{-1/4} \sum_{k=2}^{\infty} \mathrm{e}^{-T_\beta(E_k-E_0)} \|\psi_k\|_{L^1} (|\psi_k(x)|\psi_0(y) + |\psi_k(y)|\psi_0(x)) \\ + \sum_{k,j=2}^{\infty} \mathrm{e}^{-T_\beta(E_k+E_j-2E_0)} \|\psi_k\|_{L^1}\|\psi_j\|_{L^1} |\psi_k(x)||\psi_j(y)|.\end{aligned}$$

Arguing as in the proof of Lemma 3.5, $\|\psi_k\|_{L^1} \leq C\sqrt{1+\beta E_k}$ so that, by Proposition 3.1 (i), after elementary computations, we deduce that there exists a constant C such that for every $x, y \in \mathbb{R}^2$ and β large enough,

$$\begin{aligned}|\mathcal{E}_{T_\beta}(x,y)| \leq C\mathrm{e}^{-T_\beta} \sum_{k=2}^{\infty} \mathrm{e}^{-E_k} (|\psi_k(x)|\psi_0(y) + |\psi_k(y)|\psi_0(x)) \\ + C\mathrm{e}^{-2T_\beta} \sum_{k,j=2}^{\infty} \mathrm{e}^{-(E_k+E_j)} |\psi_k(x)||\psi_j(y)|.\end{aligned} \tag{3.18}$$

By Lemma 3.5 and the Cauchy–Schwarz inequality we conclude that

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \int \mathrm{d}x\,\mathrm{d}y\, |\mathcal{E}_{T_\beta}(x,y)| g_{\ell_\beta-2T_\beta}(x,y) = -\infty,$$

which, together with (3.8) and (3.16) implies the statement.

Proof of (iii). Since $\psi_1(0) = 0$, from (3.9) we deduce

$$\begin{aligned}\|\pi_\beta^{T_\beta} - \pi_\beta\|_{\mathrm{TV}} &\leq \frac{1}{2}\frac{1}{\psi_0(0)}\sum_{k=2}^{\infty} \mathrm{e}^{-T_\beta(E_k-E_0)}|\psi_k(0)|\|\psi_k\|_{L^1}\\ &\leq \frac{1}{4}\frac{1}{\psi_0(0)}\sum_{k=2}^{\infty} \mathrm{e}^{-T_\beta(E_k-E_0)}\big(|\psi_k(0)|^2 + \|\psi_k\|_{L^1}^2\big).\end{aligned}$$

Using Proposition 3.1 (iii) and Lemma 3.5, to conclude the proof of the statement it is enough to show

$$\lim_{\beta\to\infty}\frac{1}{\beta}\log\sum_{k=2}^{\infty}\mathrm{e}^{-T_\beta(E_k-E_0)}|\psi_k(0)|^2 = -\infty. \tag{3.19}$$

To this end, observe that, by Sobolev embedding and (3.1), there exists a constant $C < \infty$ such that for every k

$$\|\psi_k\|_\infty^2 \leq C\|\psi_k\|_{H^1}^2 \leq C(1 + 2\beta E_k) \tag{3.20}$$

and (3.19) follows by arguing as in the proof of Lemma 3.5. □

The last statement of this section provides a global control on $|\psi_1| - \psi_0$ in terms of the spectral gap $E_1 - E_0$. This appears to be a novel point in the semiclassical analysis of H_β and it is the crucial ingredient for obtaining the sharp asymptotics in Theorem 2.2.

Theorem 3.6. *There exists $C \in (0,\infty)$ such that for every $\beta \geq 1$*

$$\sup_{x\geq 0}\psi_0(x)\big|\psi_0(x) - \psi_1(x)\big| \leq C\beta^{\frac{3}{2}}(E_1 - E_0).$$

Proof. The proof is based on the construction of a test function f_1 which provides a good global approximation to ψ_1. To this end, let $\bar{\chi} \in C^\infty(\mathbb{R})$ be such that $0 \leq \bar{\chi} \leq 1$, $\bar{\chi}(x) = 1$ on $(-\infty, -2]$, and $\bar{\chi}(x) = 0$ on $[-1, +\infty)$. Further, for $\delta \in (0, \frac{1}{2})$, set $\chi(x) = \chi_\delta(x) = \bar{\chi}(\delta^{-1}(|x| - 1))$. Note that $\chi_\delta = 1$ on $[-1 + 2\delta, 1 - 2\delta]$ and $\chi_\delta = 0$ on the complement of $[-1 + \delta, 1 - \delta]$. Define

$$\theta(x) := \kappa\int_0^x \mathrm{d}y\,\frac{\chi(y)}{\psi_0^2(y)}, \qquad \kappa = \kappa_\beta := \left(\int_0^1 \mathrm{d}y\,\frac{\chi(y)}{\psi_0^2(y)}\right)^{-1}.$$

In particular $\theta(x) = 1$ for $x \geq 1$. Set finally

$$f_1 := \frac{1}{Z}\theta\psi_0, \qquad Z = Z_\beta := \left(\int \mathrm{d}x\,\theta^2\psi_0^2\right)^{\frac{1}{2}}.$$

Note that f_1 is antisymmetric and normalized in L^2. Further, by Proposition 3.1 (iv), for every $a \in (0, 1]$,

$$\frac{1}{\kappa} \sim \int_0^a \mathrm{d}y\,\frac{\chi(y)}{\psi_0^2(y)} \sim \int_0^a \mathrm{d}y\,\frac{1}{\psi_0^2(y)} \quad\text{and}\quad \lim_{\beta\to\infty}\frac{1}{\beta}\log\kappa = -C_{\mathrm{W}}. \tag{3.21}$$

Step 1. There exists a constant C such that, for every $\beta \geq 1$,

$$|Z-1| \leq C\kappa \quad \text{and} \quad \sup_{x \geq 0} \psi_0(x)|\psi_0(x) - f_1(x)| \leq C\sqrt{\beta}\kappa.$$

Since $E_0 \to 1$ and ψ_0 satisfies $\frac{1}{2\beta}\psi_0'' - \beta W \psi_0 = -E_0\psi_0$ we deduce that there exists a constant $c > 0$ depending only on W such that $\psi_0'(x) \geq 0$ for $x \in [0, 1 - c\beta^{-\frac{1}{2}}]$ and β large enough. Hence, choosing $\delta = \delta_\beta = c\beta^{-\frac{1}{2}}$, for $x \in [0,1]$,

$$0 \leq 1 - \theta(x) = \kappa \int_x^1 \mathrm{d}y \, \frac{\chi(y)}{\psi_0^2(y)} \leq \frac{\kappa}{\psi_0^2(x)}. \tag{3.22}$$

Therefore,

$$0 \leq 1 - Z^2 = 2\int_0^1 \mathrm{d}x \, [1 - \theta^2(x)]\psi_0^2(x) \leq 4\kappa,$$

which proves the first claim. To prove the second, we write

$$\psi_0(x)|\psi_0(x) - f_1(x)| \leq \psi_0^2(x)|1 - \theta(x)| + \psi_0^2(x)|1 - Z^{-1}|.$$

We then conclude by using (3.22), Proposition 3.1 (v), and the bound on $|Z-1|$.
Step 2. There exists a constant $C > 0$ such that for any β large enough

$$\int_{\mathbb{R}} \mathrm{d}x \, f_1(H_\beta - E_0) f_1 \sim \beta^{-1}\kappa \quad \text{and} \quad \int_{\mathbb{R}} \mathrm{d}x \, |(H_\beta - E_0) f_1|^2 \leq C\beta^{-2}\kappa^2.$$

To prove the first claim, note that by the ground state transformation,

$$\int_{\mathbb{R}} f_1(H_\beta - E_0) f_1 \mathrm{d}x = \frac{1}{2\beta}\int_{\mathbb{R}} \mathrm{d}x \left| \left(\frac{f_1}{\psi_0}\right)' \right|^2 \psi_0^2 = \frac{\kappa^2}{\beta Z^2}\int_0^1 \mathrm{d}x \, \frac{\chi^2}{\psi_0^2} \sim \beta^{-1}\kappa,$$

where we used the first statement in (3.21) and, as follows from Step 1, $Z \sim 1$. For the second claim, again by the ground state transformation,

$$\int_{\mathbb{R}} \mathrm{d}x \, \left|(H_\beta - E_0) f_1\right|^2 = \int_{\mathbb{R}} \mathrm{d}x \left| \frac{1}{2\beta}\frac{1}{\psi_0^2}\left[\psi_0^2\left(\frac{f_1}{\psi_0}\right)'\right]' \right|^2 \psi_0^2 = \frac{\kappa^2}{Z^2(2\beta)^2}\int_{\mathbb{R}} \mathrm{d}x \, \frac{|\chi'|^2}{\psi_0^2}.$$

By the definition of χ_δ, the choice of $\delta = c\beta^{-\frac{1}{2}}$, and the monotonicity of ψ_0 on the interval $[0, 1-\delta_\beta]$, we deduce that there exists a constant $C > 0$ such that, for any β large enough,

$$\int_{\mathbb{R}} \mathrm{d}x \, |(H_\beta - E_0) f_1|^2 \leq C\frac{\kappa^2}{Z^2\beta^{3/2}}\frac{1}{\psi_0^2(1-\delta_\beta)} \leq C\frac{\kappa^2}{\beta^2},$$

where we used the second bound in Proposition 3.1 (v) and $Z \sim 1$ as proven in Step 1.
Step 3. As $\beta \to \infty$, $\beta(E_1 - E_0) \sim \kappa$.

In view of the orthogonality $\langle f_1, \psi_0 \rangle = 0$ and the first statement of Step 2, the upper bound follows from the Riesz variational principle. The lower bound

follows from Step 2, $E_2 - E_0 \to 2$, as stated in Proposition 3.1 (i), and the Temple inequality, see for example [19] or [16, Prop. 5.1].
Conclusion. We write

$$\psi_0(x)|\psi_1(x) - \psi_0(x)| \le \psi_0(x)|\psi_0(x) - f_1(x)| + \psi_0(x)|\psi_1(x) - f_1(x)|.$$

The first term on the right hand side is bounded by using Steps 1 and 3. By Lemma 3.2 (ii) with $R = \infty$, $E = E_0$, $c \in (\bar{E}_0, \bar{E}_1)$, and Steps 2-3,

$$\|f_1 - \langle f_1, \psi_1\rangle \psi_1\|_{L^\infty} \le C\sqrt{\beta}(E_1 - E_0),$$

for a suitable constant C. Furthermore, since $\langle f_1, \psi_1\rangle \ge 0$, Lemma 3.2 (i) with R, E, c as before, and again Steps 2-3 imply

$$|\langle f_1, \psi_1\rangle - 1| \le |\langle f_1, \psi_1\rangle^2 - 1| \le \|f_1 - \langle f_1, \psi_1\rangle \psi_1\|_{L^2}^2 \le C(E_1 - E_0).$$

We conclude the proof by using Proposition 3.1 (v) to bound $\|\psi_k\|_\infty$ for $k = 0, 1$.
□

4. Small Noise Analysis of the Diffusion Process

The present analysis of the ϕ_1^4 measure is based on a representation of the probability $\mu_{\beta,\ell}$ in terms of the diffusion (2.15) that we here describe. Given $\ell > 0$ and $x, y \in \mathbb{R}$, denote by $\mathbb{P}_{x,y}^{\beta,\ell}$ the law of the solution to (2.15) starting at time $t = 0$ from x conditioned to $X_\ell = y$. The probability $\mathbb{P}_{x,y}^{\beta,\ell}$ can be identified with the ϕ_1^4 measure on $[0, \ell]$ with boundary conditions $X_0 = x$ and $X_\ell = y$. Since the marginal distribution of $\mu_{\beta,\ell}$ at the endpoints is the probability $\wp_{\beta,\ell}$ with density given in (3.10) with $T = 0$, we obtain the following representation

$$\mu_{\beta,\ell}(\mathrm{d}X) = \wp_{\beta,\ell}(\mathrm{d}X_0\,\mathrm{d}X_\ell)\,\mathbb{P}_{X_0,X_\ell}^{\beta,\ell}(\mathrm{d}X). \tag{4.1}$$

Given $x \in \mathbb{R}$ we denote by $\mathbb{P}_x^\beta$ the law of the solution to (2.15) with initial condition x, that we regard as a probability on $C([0, \infty))$. By using the result on the semiclassical analysis of the anharmonic oscillator deduced in Section 3, in this section we derive the asymptotic behavior of this probability in the limit $\beta \to \infty$. In the next sections, we will then achieve the proofs of the main results by exploiting the representation (4.1).

A peculiar feature of the diffusion (2.15) is that, according to Proposition 3.1 (iii), the potential U_β in (3.2) exhibits a corner at the local maximum $x = 0$ in the limit $\beta \to \infty$. This will be an important feature in the analysis of its metastable behavior, implying in particular that in the transition between the pure phases ±1 essentially no time is spent in a neighborhood of zero.

We first show that with probability super-exponentially close to one the solution to (2.15) is bounded on exponentially large intervals.

Lemma 4.1. *Let* $(\ell_\beta)_{\beta>0}$ *be such that* $\overline{\lim}_\beta\, \beta^{-1} \log \ell_\beta < \infty$. *Then for each* $L > 0$

$$\lim_{R\to\infty} \overline{\lim_{\beta\to\infty}} \sup_{|x|\le L} \frac{1}{\beta} \log \mathbb{P}_x^\beta\Big(\sup_{t\in[0,\ell_\beta]} |X_t| > R\Big) = -\infty. \tag{4.2}$$

Proof. By monotonicity we can assume $\underline{\lim}_\beta\, \beta^{-1}\ell_\beta = \infty$. For $R > 0$ let τ_R be the hitting time of $(-R, R)^{\mathrm{c}}$. We observe that $u(t, x) := \mathbb{P}_x^\beta(\tau_R > t)$ solves

$$\begin{cases} \partial_t u = \frac{1}{2\beta}\partial_x^2 u - U_\beta'(x)\partial_x u, & \text{on } (0,\infty)\times(-R,R),\\ u(0,x) = 1, & x > 0,\\ u(t,\pm R) = 0, & t > 0. \end{cases}$$

Let now $L^R = L^R_\beta$ be the selfadjoint realization on $L^2((-R, R), \pi_\beta)$ of $\frac{1}{2\beta}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - U_\beta'(x)\frac{\mathrm{d}}{\mathrm{d}x}$ with zero Dirichlet boundary condition in $x = \pm R$. Here, we understand that π_β is the sub-probability on $(-R, R)$ given by $\psi_0^2\mathrm{d}x$. Analogously, we let $H^R = H^R_\beta$ be the selfadjoint realization on $L^2((-R, R))$ of the Schrödinger operator $-\frac{1}{2\beta}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta W$ with zero Dirichlet boundary condition in $x = \pm R$. Let also $(E_k^R)_{k\ge 0}$ be the eigenvalues of H^R and let $(\psi_k^R)_{k\ge 0}$ be a corresponding orthonormal basis of eigenfunctions. By the ground state transformation, $L^R = -\frac{1}{\psi_0}(H^R - E_0)\psi_0$, so that

$$u(t,x) = \frac{1}{\psi_0(x)}\sum_{k=0}^{\infty} \mathrm{e}^{-t(E_k^R - E_0)}\langle\psi_0, \psi_k^R\rangle_{L^2((-R,R))}\psi_k^R(x).$$

Hence, by the antisymmetry of ψ_1^R,

$$\begin{aligned} \mathbb{P}_x^\beta\Big(\sup_{t\in[0,\ell_\beta]}|X_t| > R\Big) &= 1 - u(\ell_\beta, x)\\ &= \frac{1}{\psi_0(x)}\Big(\psi_0(x) - \mathrm{e}^{-\ell_\beta(E_0^R - E_0)}\langle\psi_0,\psi_0^R\rangle_{L^2((-R,R))}\psi_0^R(x)\Big)\\ &\quad + \frac{1}{\psi_0(x)}\sum_{k=2}^{\infty}\mathrm{e}^{-\ell_\beta(E_k^R - E_0)}\langle\psi_0,\psi_k^R\rangle_{L^2((-R,R))}\psi_k^R(x). \end{aligned}$$

By Proposition 3.1 (iii) and (3.4), for each $L > 0$,

$$\lim_\beta \sup_{|x|\le L}\frac{1}{\beta}\log\frac{1}{\psi_0(x)} = \max_{|x|\le L} U(x) < \infty.$$

The proof is thus completed provided we show that, for each $L > 0$,

$$\lim_{R\to\infty}\overline{\lim_{\beta\to\infty}}\frac{1}{\beta}\log(E_0^R - E_0) = -\infty, \tag{4.3}$$

$$\lim_{R\to\infty}\overline{\lim_{\beta\to\infty}}\sup_{|x|\le L}\frac{1}{\beta}\log\left|\psi_0(x) - \langle\psi_0,\psi_0^R\rangle_{L^2((-R,R))}\psi_0^R(x)\right| = -\infty, \tag{4.4}$$

$$\lim_{\beta\to\infty}\sup_{|x|\le L}\frac{1}{\beta}\log\sum_{k\ge 2}\mathrm{e}^{-\ell_\beta(E_k^R - E_0)}|\psi_k^R(x)| = -\infty. \tag{4.5}$$

To prove (4.3), let $\chi_R \in C_c^\infty((-R,R);[0,1])$ be symmetric and such that $\chi_R(x)=1$ for $|x|\le R-1$ and $|\chi_R'(x)|\le 2$. Observe that

$$\langle \chi_R\psi_0, (H^R-E_0)\chi_R\psi_0\rangle = \frac{1}{2\beta}\int \mathrm{d}x\, |\chi_R'|^2\psi_0^2.$$

The claim then follows from Proposition 3.1 (iv) and the variational characterization of the bottom of the spectrum.

In order to prove (4.4), we set $f^R=\psi_0\chi_R$ with χ_R as above. Choosing $c\in(E_0^R,E_2^R)$ and using the antisymmetry of ψ_1^R,

$$g^R := \mathbf{1}_{[c,\infty)}(H^R)f^R = f^R - \langle f^R,\psi_0^R\rangle_{L^2((-R,R))}\psi_0^R.$$

We then have, for R large enough,

$$\begin{aligned}
&\sup_{|x|\le L}\left|\psi_0(x)-\langle\psi_0,\psi_0^R\rangle_{L^2((-R,R))}\psi_0^R(x)\right|\\
&\quad=\sup_{|x|\le L}\left|f^R(x)-\langle\psi_0,\psi_0^R\rangle_{L^2((-R,R))}\psi_0^R(x)\right|\\
&\quad\le\sup_{|x|\le R}|g^R(x)|+|\langle\psi_0(\chi_R-1),\psi_0^R\rangle_{L^2((-R,R))}|\sup_{|x|\le R}\psi_0^R(x)\\
&\quad\le\sup_{|x|\le R}|g^R(x)|+2\|\psi_0^R\|^2_{H_0^1((-R,R))}\int_{R-1}^\infty \mathrm{d}x\,\psi_0(x).
\end{aligned}$$

Observing that $\|\psi_0^R\|^2_{H_0^1((-R,R))}\le 2\beta E_0^R+1$ and using that $E_0^R\to\bar E_0$ as $\beta\to\infty$, the second term is super-exponentially small in view of Proposition 3.1 (iv). To bound the first term we observe that $E_2^R\to\bar E_1$ and apply Lemma 3.2 (ii) with $E=E_0$ and $c\in(\bar E_0,\bar E_1)$. We deduce that there exists a constant C such that for any β large enough

$$\sup_{|x|\le R}|g^R(x)|^2\le C\beta\|(H^R-E_0)f^R\|^2_{L^2((-R,R))}.$$

The norm on the right hand side reads

$$\begin{aligned}
&\|(H^R-E_0)f^R\|^2_{L^2((-R,R))}\\
&\quad=\int_{-R}^R \mathrm{d}x\left[-\frac{1}{2\beta}(\chi_R\psi_0)''+(\beta W-E_0)\chi_R\psi_0\right]^2\\
&\quad=\int_{-R}^R \mathrm{d}x\left(\frac{1}{2\beta}\chi_R''\psi_0+\frac{1}{\beta}\chi_R'\psi_0'\right)^2\\
&\quad\le\frac{C}{\beta^2}\left(\int_{-R}^R \mathrm{d}x\,(\chi_R'')^2\psi_0^2+\int_{-R}^R \mathrm{d}x\,(\chi_R')^2(\psi_0')^2\right).
\end{aligned}$$

The first term in the right-hand side is directly estimated by using Proposition 3.1 (iv). For the second one, we integrate by parts, obtaining

$$\int_{-R}^{R} \mathrm{d}x\, (\chi_R')^2 (\psi_0')^2 = -\int_{-R}^{R} \mathrm{d}x \left[((\chi_R')^2)' \psi_0' \psi_0 + (\chi_R')^2 \psi_0'' \psi_0 \right]$$
$$= \int_{-R}^{R} \mathrm{d}x \left[\frac{1}{2} ((\chi_R')^2)'' \psi_0^2 - (\chi_R')^2 \psi_0'' \psi_0 \right].$$

We conclude by using $-\frac{1}{2\beta}\psi_0'' + \beta W \psi_0 = E_0 \psi_0$ and again Proposition 3.1 (iv).

To prove (4.5), we first observe that by the Sobolev embedding

$$\sup_{|x|\le L} |\psi_k^R(x)|^2 \le \sup_{|x|\le R} |\psi_k^R(x)|^2 \le \|\psi_k^R\|_{H^1}^2 \le \big(1 + 2\beta(E_k^R - E_0)\big),$$

so that

$$\sup_{|x|\le L} \sum_{k=2}^{\infty} \mathrm{e}^{-\ell_\beta (E_k^R - E_0)} |\psi_k^R(x)| \le \sum_{k=2}^{\infty} \mathrm{e}^{-\ell_\beta (E_k^R - E_0) + 2\beta (E_k^R - E_0)}.$$

Recalling we have assumed $\underline{\lim}_\beta\, \beta^{-1}\ell_\beta = \infty$, by Weyl's Theorem [23], the min-max principle, $E_k^R \ge E_k$, and Proposition 3.1 (iii) we conclude by arguing as in the proof of Lemma 3.5. □

We next show that, with $\mathbb{P}_x^\beta$ probability super-exponentially close to one, the time average of $t \mapsto W(X_t)$ is arbitrarily small. For the analysis of the sharp asymptotics, we need a quantitative statement that requires condition (2.11).

Lemma 4.2. *(i) Fix* $\delta > 0$ *and a sequence* $(\ell_\beta)_{\beta>0}$ *such that* $\lim_\beta \ell_\beta = \infty$. *Then, for any* $L > 0$,

$$\lim_{\beta\to\infty} \frac{1}{\beta} \sup_{|x|\le L} \log \mathbb{P}_x^\beta \left(\frac{1}{\ell_\beta} \int_0^{\ell_\beta} \mathrm{d}t\, W(X_t) > \delta \right) = -\infty.$$

(ii) Fix $\gamma > \bar{E}_0 = 1$ *and a sequence* $(\ell_\beta)_{\beta>0}$ *such that* $\lim_\beta \beta^{-2}\ell_\beta = \infty$. *Then, for any* $L > 0$,

$$\lim_{\beta\to\infty} \frac{1}{\beta} \sup_{|x|\le L} \log \mathbb{P}_x^\beta \left(\frac{1}{\ell_\beta} \int_0^{\ell_\beta} \mathrm{d}t\, W(X_t) > \frac{\gamma}{\beta} \right) = -\infty.$$

Proof. Recalling the definition (3.2) of U_β, by Itô's formula,

$$U_\beta(X_t) - U_\beta(x) + \int_0^t \mathrm{d}r \left(U_\beta'(X_r)^2 - \frac{1}{2\beta} U_\beta''(X_r) \right) = M_t^\beta, \tag{4.6}$$

where $(M_t^\beta)_{t\ge 0}$ is a continuous $\mathbb{P}_x^\beta$-martingale with quadratic variation

$$\langle M^\beta \rangle_t = \frac{1}{\beta} \int_0^t \mathrm{d}r\, U_\beta'(X_r)^2.$$

We claim that there exists $C \in (0,\infty)$ and for any $L > 0$ a constant C_L such that for any β large enough

$$\frac{1}{2\beta} U_\beta'' \le \frac{1}{2} (U_\beta')^2 + C, \qquad U_\beta \ge -C, \qquad \max_{|x|\le L} U_\beta(x) \le C_L.$$

Indeed, the first bound follows directly from Proposition 3.1 (i), Riccati's equation (3.3), and the positivity of W. The second one follows from items (iii) and (iv) in Proposition 3.1 and the last statement again from Proposition 3.1 (iii).

Equation (4.6) thus implies that for each $x \in [-L, L]$ with $\mathbb{P}_x^\beta$ probability one we have

$$\frac{\beta}{2}\langle M^\beta \rangle_t \le M_t^\beta + C(t+1) + C_L, \quad t \ge 0.$$

By using the generalization of the well known Bernstein inequality given in [29, Lemma 2], we deduce that, for any $\eta, \ell > 0$,

$$\sup_{|x| \le L} \mathbb{P}_x^\beta\Big(\sup_{t \in [0,\ell]} M_t^\beta > \eta \ell \Big) \le \exp\Big\{ -\frac{\beta \eta^2 \ell}{4(\eta + C) + 4(C + C_L)\ell^{-1}} \Big\}. \tag{4.7}$$

We next observe that by (3.3) and (4.6)

$$\begin{aligned}\int_0^\ell \mathrm{d}t\, W(X_t) &= \int_0^\ell \mathrm{d}t \left(\frac{1}{2} U_\beta'(X_t)^2 - \frac{1}{2\beta} U_\beta''(X_t) + \frac{1}{\beta} E_{0,\beta} \right) \\ &\le M_\ell^\beta - \big(U_\beta(X_\ell) - U_\beta(x) \big) + \frac{1}{\beta} E_{0,\beta} \le M_\ell^\beta + C + C_L + \frac{1}{\beta} E_{0,\beta} \ell.\end{aligned}$$

Recalling that by Proposition 3.1 (i) $E_{0,\beta} \to \bar{E}_0$, the proof of both statements is achieved by combining the previous displayed bound with (4.7). □

The next statement will be used in the proof of the lower bound in Theorem 2.1.

Lemma 4.3. *Fix $\delta \in (0, 1)$ and set $I_\delta^\pm := (\pm 1 - \delta, \pm 1 + \delta)$. There exists $\gamma > 0$ such that*

$$\lim_{\beta \to \infty} \mathbb{P}_y^\beta (X_t \in I_\delta^\pm) = 1 \quad \textit{uniformly for } y \in I_\delta^\pm \textit{ and } t \in [\gamma\beta, \ell_\beta].$$

Proof. Using the antisymmetry of $\psi_{1,\beta}$ we have,

$$\begin{aligned}&\mathbb{P}_y^\beta (X_t \in I_\delta^\pm) \\ &\quad = \frac{1}{\psi_{0,\beta}(y)} \int_{I_\delta^\pm} \mathrm{d}x\, g(x, y; t) \psi_{0,\beta}(x) \\ &\quad = \int_{I_\delta^\pm} \mathrm{d}x\, \psi_{0,\beta}(x)^2 + \mathrm{e}^{-t(E_{1,\beta} - E_{0,\beta})} \frac{|\psi_{1,\beta}(y)|}{\psi_{0,\beta}(y)} \\ &\quad\quad \times \int_{I_\delta^\pm} \mathrm{d}x\, \psi_{0,\beta}(x) |\psi_{1,\beta}(x)| + R_\beta(y, t),\end{aligned}$$

where

$$R_\beta(y, t) = \sum_{k=2}^\infty \mathrm{e}^{-t(E_{k,\beta} - E_{0,\beta})} \frac{\psi_{k,\beta}(y)}{\psi_{0,\beta}(y)} \int_{I_\delta^\pm} \mathrm{d}x\, \psi_{0,\beta}(x) \psi_{k,\beta}(x)$$

and

$$\frac{|\psi_{1,\beta}(y)|}{\psi_{0,\beta}(y)} = 1 + \frac{\psi_{0,\beta}(y)(|\psi_{1,\beta}(y)| - \psi_{0,\beta}(y))}{\psi_{0,\beta}^2(y)}.$$

By Theorem 3.6 and Proposition 3.1 (iii),

$$\sup_{y \in I_\delta^\pm} \left| \frac{\psi_{0,\beta}(y)(|\psi_{1,\beta}(y)| - \psi_{0,\beta}(y))}{\psi_{0,\beta}^2(y)} \right| \le C\beta(E_{1,\beta} - E_{0,\beta}) \exp\Big\{2\beta \max_{y \in I_\delta^\pm} U(y)\Big\},$$

which vanishes as $\beta \to \infty$. Moreover, by Proposition 3.1 (ii), $\mathrm{e}^{-t(E_{1,\beta}-E_{0,\beta})} \to 1$ for $t \le \ell_\beta$ as $\beta \to \infty$ and, by (3.7),

$$\lim_{\beta\to\infty} \int_{I_\delta^\pm} \mathrm{d}x\, \psi_{0,\beta}(x)^2 = \lim_{\beta\to\infty} \int_{I_\delta^\pm} \mathrm{d}x\, \psi_{0,\beta}(x)|\psi_{1,\beta}(x)| = \frac{1}{2}.$$

Finally, in view of Proposition 3.1 (iii) and (3.20), arguing as in Lemma 3.5,

$$\begin{aligned} \max_{y\in I_\delta^\pm} |R_\beta(y,t)| &\le C\mathrm{e}^{C_\delta \beta} \sum_{k=2}^\infty (1+\beta E_{k,\beta})\mathrm{e}^{-t(E_{k,\beta}-E_{0,\beta})} \\ &\le C \exp\Big\{2\beta \max_{y\in I_\delta^\pm} U(y)\Big\} \mathrm{e}^{-(t-\beta)(E_{2,\beta}-E_{0,\beta})}, \end{aligned}$$

which vanishes provided γ is large enough. □

For $T > 0$ let $\mathbb{P}_x^{\beta,T}$ be the marginal of $\mathbb{P}_x^\beta$ on $C([0,T])$. In view of the singularity of the potential in the limit $\beta \to \infty$, the diffusion (2.15) does not fit in the standard Freidlin-Wentzell scheme [17]. Nonetheless, as we next prove, $\mathbb{P}_x^{\beta,T}$ satisfies a large deviation principle with rate function $I_T^x \colon C([0,T]) \to [0,\infty]$ given by

$$I_T^x(X) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \mathrm{d}t\, \big[\dot X_t^2 + U'(X_t)^2\big] + U(X_T) - U(X_0) & \text{if } X \in H_x^1([0,T]), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.8}$$

where

$$H_x^1([0,T]) := \Big\{X \in C([0,T]) \colon \int_0^T \mathrm{d}t\, \dot X_t^2 < \infty, \quad X_0 = x\Big\}$$

and U has been defined in (3.4). We emphasize that I_T^x is well defined because $(U')^2$ is continuous. Note also that, informally, I_T^x is equal to the standard, but ill-defined, Friedlin-Wentzell rate function $(1/2)\int_0^T \mathrm{d}t\, \big[\dot X_t - U'(X_t)\big]^2$.

Proposition 4.4. *Fix $T > 0$. The sequence $(\mathbb{P}_x^{\beta,T})_{\beta>0}$ satisfies, uniformly for x in compact sets, a large deviation principle with speed β and rate function I_T^x. Namely, for each $x \in \mathbb{R}$, each sequence $x_\beta \to x$, each closed $\mathcal{C} \subset C([0,T])$, and each open $\mathcal{O} \subset C([0,T])$,*

$$\varlimsup_{\beta\to\infty} \frac{1}{\beta} \log \mathbb{P}_{x_\beta}^{\beta,T}(\mathcal{C}) \le -\inf_{X\in\mathcal{C}} I_T^x(X), \qquad \varliminf_{\beta\to\infty} \frac{1}{\beta} \log \mathbb{P}_{x_\beta}^{\beta,T}(\mathcal{O}) \ge -\inf_{X\in\mathcal{O}} I_T^x(X).$$

Proof. Let $\mathbb{P}_x^{0,\beta,T}$ be the law of $x+\beta^{-1/2}\,w$ on $C([0,T])$. By the Schilder theorem the sequence $(\mathbb{P}_x^{0,\beta,T})_{\beta>0}$ satisfies, uniformly for x in compact sets, a large deviation principle with speed β and rate function

$$J_T^x(X)=\begin{cases}\dfrac{1}{2}\displaystyle\int_0^T \mathrm{d}t\,\dot{X}_t^2 & \text{if } X\in H_x^1([0,T]),\\ +\infty & \text{otherwise.}\end{cases}$$

Let $L=L_\beta=\frac{1}{2\beta}\frac{\mathrm{d}^2}{\mathrm{d}x^2}-U_\beta'\frac{\mathrm{d}}{\mathrm{d}x}$ be the generator of (2.15) and denote by $p_t^\beta(x,y)$, $t>0$, $x,y\in\mathbb{R}$, the kernel of the semigroup generated by L. Recalling that $g_t^\beta(x,y)$ is the kernel of the semigroup generated by $-(H_\beta-E_{0,\beta})$, the ground state transformation yields $p_t^\beta(x,y)=\psi_{0,\beta}(x)^{-1}g_t^\beta(x,y)\psi_{0,\beta}(y)$. By the Feynman-Kac representation for the kernel $g_t^\beta(x,y)$ we deduce

$$\frac{\mathrm{d}\mathbb{P}_x^{\beta,T}}{\mathrm{d}\mathbb{P}_x^{0,\beta,T}}=\exp\big\{-\beta\Phi_\beta^x\big\},\tag{4.9}$$

in which

$$\Phi_\beta^x(X)=\int_0^T\mathrm{d}t\,\Big[W(X_t)-\frac{1}{\beta}E_{0,\beta}\Big]+U_\beta(X_T)-U_\beta(x).$$

By items (i) and (iii) in Proposition 3.1, Φ_β^x converges uniformly on compact subsets of $C([0,T])$ to

$$\begin{aligned}\Phi^x(X)&:=\int_0^T\mathrm{d}t\,W(X_t)+U(X_T)-U(x)\\&=\frac{1}{2}\int_0^T\mathrm{d}t\,U'(X_t)^2+U(X_T)-U(x),\end{aligned}$$

uniformly for x in compact subsets of $\mathbb{R}$.

The proof is now completed by applying the corollary of the Laplace–Varadhan principle stated in [13, Ex. 4.3.11]. Observe indeed that the latter can be generalized to the present setting in which Φ_β^x depends on β, but converges uniformly in compacts. It remains to show the tail condition [13, 4.3.2], that in view of (4.9) can be restated as

$$\lim_{R\to\infty}\varlimsup_{\beta\to\infty}\sup_{|x|\le L}\frac{1}{\beta}\log\mathbb{P}_x^{\beta,T}\big(\|X\|_\infty>R\big)=-\infty,\quad L>0,\tag{4.10}$$

and therefore follows directly from Lemma 4.1. □

The next statement provides the sharp asymptotics of the hitting time of zero for the stationary process associated to (2.15). It is the key ingredient to derive the sharp asymptotics of the ϕ_1^4 measure described in Theorem 2.2. Recall that $\pi_\beta(\mathrm{d}x)=\psi_0^2\mathrm{d}x$ is the invariant measure of the diffusion (2.15). We denote by $\mathbb{P}_{\pi_\beta}^\beta$ the law of the corresponding stationary process.

Theorem 4.5. *Let* τ *be the hitting time of zero and* $(T_{k,\beta})_{\beta>0}$*,* $k=0,1$*, be sequences such that* $\lim_\beta \beta^{-3/2}T_{0,\beta}=\infty$ *and* $\lim_\beta \bar{\ell}_\beta^{-1}T_{1,\beta}=0$*. Then, uniformly for* $t\in[T_{0,\beta},T_{1,\beta}]$,

$$\mathbb{P}^{\beta}_{\pi_\beta}(\tau<t)\sim\frac{2t}{\bar{\ell}_\beta}=tA_{\mathrm{W}}\sqrt{\beta}\mathrm{e}^{-\beta C_{\mathrm{W}}}.$$

Proof. By symmetry,

$$\mathbb{P}^{\beta}_{\pi_\beta}(\tau<t)=1-2\int_0^\infty \mathrm{d}x\,\psi_0^2(x)u(t,x),$$

where $u(t,x):=\mathbb{P}^\beta_x(\tau>t)$ solves

$$\begin{cases}\partial_t u=\frac{1}{2\beta}\partial_x^2u-U'_\beta\partial_x u, & \text{on } (0,\infty)\times(0,\infty),\\ u(0,x)=1, & x>0,\\ u(t,0)=0, & t>0.\end{cases}$$

Denote by L^0_β the selfadjoint realization on $L^2(\mathbb{R}_+,2\pi_\beta)$ of $\frac{1}{2\beta}\frac{\mathrm{d}^2}{\mathrm{d}x^2}-U'_\beta\frac{\mathrm{d}}{\mathrm{d}x}$ with zero Dirichlet boundary condition in $x=0$. By symmetry and the ground state transformation, the eigenvalues of L^0_β are $-\lambda_k=E_{2k-1,\beta}-E_{0,\beta}$, $k\geq 1$, and the corresponding normalized eigenfunctions are $f_k=\psi_{2k-1}/\psi_0$, $k\geq 1$. Hence

$$u(t,x)=\sum_{k=1}^\infty e^{-t\lambda_k}\langle f_k,1\rangle_{L^2(\mathbb{R}_+,2\pi_\beta)}f_k(x).$$

By Cauchy–Schwarz inequality and Lemma 3.5 there exists a constant C such that, for β large enough,

$$\sum_{k=2}^\infty \mathrm{e}^{-t\lambda_k}|\langle f_k,1\rangle_{L^2(\mathbb{R}_+,2\pi_\beta)}|\int_0^\infty \mathrm{d}x\,|f_k(x)|\psi_0^2(x)\leq C\mathrm{e}^{-t},$$

so that

$$\left|\mathbb{P}^\beta_{\pi_\beta}(\tau<t)-\left(1-2\mathrm{e}^{-t\lambda_1}\langle f_1,1\rangle_{L^2(\mathbb{R}_+,2\pi_\beta)}\int_0^\infty \mathrm{d}x\,f_1(x)\psi_0^2(x)\right)\right|\leq C\mathrm{e}^{-t}. \tag{4.11}$$

On the other hand,

$$\begin{aligned}&1-2\mathrm{e}^{-t\lambda_1}\langle f_1,1\rangle_{L^2(\mathbb{R}_+,2\pi_\beta)}\int_0^\infty \mathrm{d}x\,f_1(x)\psi_0^2(x)\\ &\quad=1-4\mathrm{e}^{-t(E_{1,\beta}-E_{0,\beta})}\left(\int_0^\infty \mathrm{d}x\,\psi_1(x)\psi_0(x)\right)^2\\ &\quad=1-4\mathrm{e}^{-t(E_{1,\beta}-E_{0,\beta})}\left(\frac{1}{2}+\int_0^\infty \mathrm{d}x\,[\psi_1(x)-\psi_0(x)]\psi_0(x)\right)^2.\end{aligned}$$

We conclude by using items (ii) and (iv) in Proposition 3.1, and Theorem 3.6. ◻

We finally show that under the measure $\mathbb{P}^{\beta}_{\pi_\beta}$ the random variable $\tau/\bar{\ell}_\beta$ converges in law to an exponential random variable with parameter 2.

Theorem 4.6. *Let τ be the hitting time of zero. Then, for each $t \geq 0$,*

$$\lim_{\beta\to\infty} \mathbb{P}^{\beta}_{\pi_\beta}\Big(\frac{\tau}{\bar{\ell}_\beta} > t\Big) = \mathrm{e}^{-2t}.$$

Proof. We use the notation introduced in the proof of Theorem 4.5. As follows from (4.11), for $t > 0$,

$$\lim_{\beta\to\infty} \bigg| \mathbb{P}^{\beta}_{\pi_\beta}(\tau > t\bar{\ell}_\beta) - 2\mathrm{e}^{-t\bar{\ell}_\beta\lambda_1} \langle f_1, 1\rangle_{L^2(\mathbb{R}_+, 2\pi_\beta)} \int_0^\infty \mathrm{d}x\, f_1(x)\psi_0^2(x)\bigg)\bigg| = 0.$$

By (2.6) and Proposition 3.1 (ii), $\bar{\ell}_\beta\lambda_1 = \bar{\ell}_\beta(E_{1,\beta} - E_{0,\beta}) \to 2$. Moreover, by using Proposition 3.1 (iv) and Theorem 3.6,

$$\begin{aligned}
&\lim_{\beta\to\infty} 2\langle f_1, 1\rangle_{L^2(\mathbb{R}_+, 2\pi_\beta)} \int_0^\infty \mathrm{d}x\, f_1(x)\psi_0^2(x) \\
&\quad = \lim_{\beta\to\infty} 4\bigg(\int_0^\infty \mathrm{d}x\, \psi_1(x)\psi_0(x)\bigg)^2 = 1,
\end{aligned}$$

which concludes the proof. □

5. Large Deviation Principle

In this section we complete the proof of Theorem 2.1 by following the classical strategy of large deviation estimates. Namely, we first show that the family $(\mu_\beta)_{\beta>0}$, as defined in (2.9), is exponentially tight, then we prove the upper bound on compacts, and finally derive the lower bound for neighborhoods of elements X such that $I(X)$, as defined in (2.10), is finite. Some of the bounds here obtained are in fact weaker than the sharp bound in Theorem 4.5, but they rely on the assumption (2.8) rather than (2.11). Within this section we fix $p \in [1, \infty)$.

Super-Exponential Estimates

Recall that $\imath_\beta$ denotes both the dilation $(0, \ell_\beta) \ni t \mapsto s = t/\ell_\beta \in (0, 1)$ and its lift to functions, that is, $(\imath_\beta X)(s) = X(\ell_\beta s)$, $s \in (0, 1)$. According to the notation in Section 4, $\mathbb{P}^{\beta,\ell_\beta}_x$ is the law of the diffusion process satisfying (2.15) on $[0, \ell_\beta]$ with initial condition $x \in \mathbb{R}$, while μ_{β,ℓ_β} is the ϕ^4_1 measure on the interval $[0, \ell_\beta]$ with free boundary conditions on the end points. We regard both these measures as probabilities on $C([0, \ell_\beta])$. We first show that events which have super-exponentially small probability with respect to $\mathbb{P}^{\beta,\ell_\beta}_x$ have also super-exponentially small probability with respect to μ_{β,ℓ_β}.

Lemma 5.1. *Let* $(\mathcal{A}_{\beta,k})_{k\geq 1}$ *be a family of Borel sets in* $C([0,\ell_\beta])$ *such that for each* $L>0$

$$\lim_{k\to\infty}\varlimsup_{\beta\to\infty}\sup_{|x|\leq L}\frac{1}{\beta}\log\mathbb{P}_x^{\beta,\ell_\beta}\big(\mathcal{A}_{\beta,k}\big)=-\infty.$$

Then

$$\lim_{k\to\infty}\varlimsup_{\beta\to\infty}\frac{1}{\beta}\log\mu_{\beta,\ell_\beta}\big(\mathcal{A}_{\beta,k}\big)=-\infty.$$

Proof. By the representation (4.1) and Proposition 3.4 (i), there exists $C_L\to\infty$ as $L\to\infty$ such that

$$\mu_{\beta,\ell_\beta}\big(\mathcal{A}_{\beta,k}\big)\leq\int_{[-L,L]^2}\wp_\beta(\mathrm{d}x,\mathrm{d}y)\,\mathbb{P}_{x,y}^{\beta,\ell_\beta}\big(\mathcal{A}_{\beta,k}\big)+\mathrm{e}^{-\beta C_L}.$$

By the ground state transformation

$$\wp_\beta(\mathrm{d}x\,\mathrm{d}y)=\frac{1}{Z_\beta}\frac{\psi_0(x)}{\psi_0(y)}\,\mathbb{P}_x^{\beta,\ell_\beta}(X_{\ell_\beta}\in\mathrm{d}y)\,\mathrm{d}x,\tag{5.1}$$

with $Z_\beta=\int\mathrm{d}a\,\mathrm{d}b\,g_{\ell_\beta}(a,b)$. We thus deduce

$$\mu_{\beta,\ell_\beta}\big(\mathcal{A}_{\beta,k}\big)\leq\frac{1}{Z_\beta}\sup_{|y|\leq L}\frac{1}{\psi_0(y)}\sup_{|x|\leq L}\mathbb{P}_x^{\beta,\ell_\beta}\big(\mathcal{A}_{\beta,k}\big)\,\|\psi_0\|_{L^1}+\mathrm{e}^{-\beta C_L}.$$

The statement now follows from (3.16), Proposition 3.1 (iii), and Corollary 3.3. □

As we next show, the marginal of the probability μ_{β,ℓ_β} on $C([T,\ell_\beta-T])$ is super-exponentially close to $\mathbb{P}^\beta_{\pi_\beta}$ provided $T\gg\beta$. For the application to the sharp asymptotics, we also need an analogous statement for the probability $\mathbb{P}^\beta_0$, the law of the diffusion (2.15) starting from zero. We emphasize that these estimates depend on the symmetry with respect to the map $X\mapsto -X$.

Lemma 5.2. *Let* $(T_\beta)_{\beta>0}$, $(\ell_\beta)_{\beta>0}$ *be sequences such that* $\beta^{-1}T_\beta\to\infty$ *and* $\ell_\beta>2T_\beta$. *Then*

$$\lim_{\beta\to\infty}\frac{1}{\beta}\log\big|\mu_{\beta,\ell_\beta}(\mathcal{B})-\mathbb{P}^\beta_{\pi_\beta}(\mathcal{B})\big|=-\infty,\qquad\mathcal{B}\in\sigma(\{X_t\}_{t\in[T_\beta,\ell_\beta-T_\beta]}),$$
$$\lim_{\beta\to\infty}\frac{1}{\beta}\log\big|\mathbb{P}^\beta_0(\mathcal{B})-\mathbb{P}^\beta_{\pi_\beta}(\mathcal{B})\big|=-\infty,\qquad\mathcal{B}\in\sigma(\{X_t\}_{t\geq T_\beta}).$$

Proof. For notational convenience, we set $\ell=\ell_\beta$ and $T=T_\beta$. Recalling that $\bar{\wp}^T=\bar{\wp}^T_{\beta,\ell}$ is the probability measure on $\mathbb{R}^2$ with density $\bar{\varrho}^T$ defined in (3.11), we claim that

$$\mathbb{P}^\beta_{\pi_\beta}(\mathcal{B})=\int\bar{\wp}^T(\mathrm{d}x\,\mathrm{d}y)\,\mathbb{P}_{x,y}^{\beta,\ell-2T}\big(\theta_{-T}\mathcal{B}\big),$$

where θ_t is the time shift $(\theta_t X)_{t'} = X_{t'-t}$. Indeed, by the ground state transformation, $\mathbb{P}^\beta_x(X_t \in \mathrm{d}y) = \psi_0(x)^{-1} g_t(x, y)\psi_0(y)\mathrm{d}y$. By the Markov property and the explicit form (3.11) we deduce the claim.

On the other hand, recalling that $\wp^T = \wp^T_{\beta,\ell}$ is the probability measure on $\mathbb{R}^2$ with density ϱ^T defined in (3.10),

$$\mu_{\beta,\ell}(\mathcal{B}) = \int \wp^T(\mathrm{d}x\,\mathrm{d}y)\,\mathbb{P}^{\beta,\ell-2T}_{x,y}\big(\theta_{-T}\mathcal{B}\big).$$

The first statement thus follows from Proposition 3.4 (ii).

To prove the second statement, recall that π^T is the probability on $\mathbb{R}$ with density defined in (3.12). In particular, again by the ground state transformation and the Markov property,

$$\mathbb{P}^\beta_0(\mathcal{B}) = \int \pi^T(\mathrm{d}x)\,\mathbb{P}^\beta_x\big(\theta_{-T}\mathcal{B}\big).$$

The second statement thus follows from Proposition 3.4 (iii). □

Exponential Tightness

Within the general strategy of [17], we next introduce a discrete time Markov chain that allows to reduce the analysis of the asymptotic behavior of μ_{β,ℓ_β} to that of its marginal on sub-intervals of $[0, \ell_\beta]$ of fixed size. Given $\rho = (\rho_1, \rho_2)$ with $0 < \rho_1 < \rho_2 < 1$ and $X \in C([0, \ell_\beta])$, we recursively define the sequence of stopping times $(\sigma_k)_{k\in\mathbb{Z}_+}$ by

$$\begin{aligned}\sigma_{2k} &= \inf\{t \in [\sigma_{2k-1}, \ell_\beta] \colon X_t \in \{-1+\rho_1, 1-\rho_1\}\} \wedge \ell_\beta,\\ \sigma_{2k+1} &= \inf\{t \in [\sigma_{2k}, \ell_\beta] \colon X_t \in \{-1+\rho_2, 1-\rho_2\}\} \wedge \ell_\beta,\end{aligned} \tag{5.2}$$

where it is understood $\sigma_{-1} = 0$, see Figure 1. Moreover, let $\mathcal{N}_{\rho,\beta} = \sup\{k \geq 0 \colon \sigma_k < \ell_\beta\}$ so that $\sigma_k = \ell_\beta$ for $k > \mathcal{N}_{\rho,\beta}$. We then set $Y_k = X_{\sigma_k}$ and consider the family $(Y_k)_{k=0}^{\mathcal{N}_{\rho,\beta}}$ taking values in the set $\{-1+\rho_1, -1+\rho_2, 1-\rho_2, 1-\rho_1\}$.

By the strong Markov property, when X is sampled according to $\mathbb{P}^{\beta,\ell_\beta}_x$, the family (Y_k) is a homogeneous discrete time Markov chain with transition probability given by

$$\begin{pmatrix} 0 & 1 & 0 & 0\\ 1-p & 0 & p & 0\\ 0 & p & 0 & 1-p\\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where, denoting by τ_z the hitting time of z,

$$p = p_{\rho,\beta} = \mathbb{P}^\beta_{-1+\rho_2}(\tau_{1-\rho_1} < \tau_{-1+\rho_1}) = \mathbb{P}^\beta_{1-\rho_2}(\tau_{-1+\rho_1} < \tau_{1-\rho_1}).$$

We next show that the number of jumps of this chain is bounded by ℓ_β with probability super-exponentially close to one.

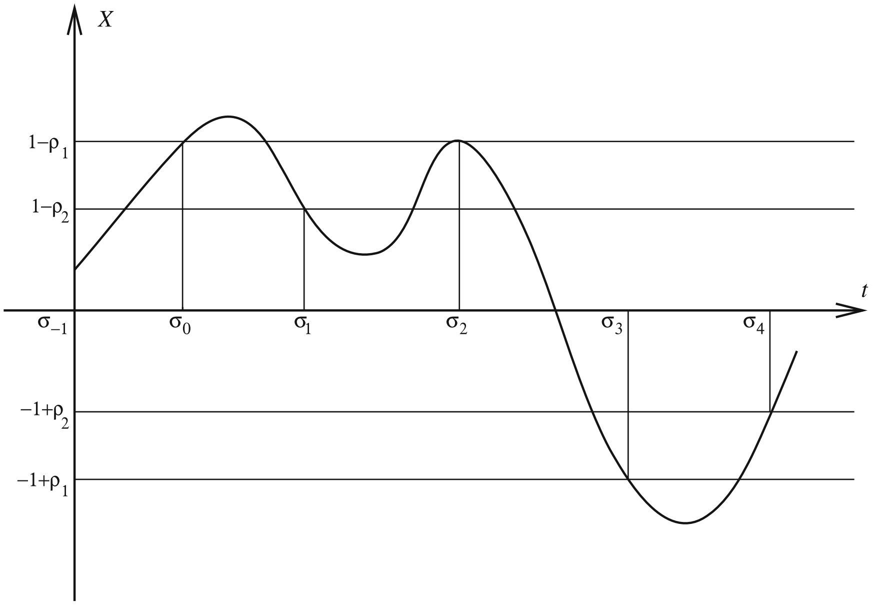

Fig. 1. An example of stopping times $\sigma_k < \ell_\beta$

Lemma 5.3. *For any* $\rho = (\rho_1, \rho_2)$ *with* $0 < \rho_1 < \rho_2 < 1$,

$$\varlimsup_{\beta\to\infty} \sup_{x\in\mathbb{R}} \frac{1}{\beta} \log \mathbb{P}_x^{\beta,\ell_\beta}(\mathcal{N}_{\rho,\beta} > \ell_\beta) = -\infty.$$

Proof. By the exponential Chebyshev inequality, for every $\gamma > 0$,

$$\begin{aligned}\mathbb{P}_x^{\beta,\ell_\beta}(\mathcal{N}_{\rho,\beta} > \ell_\beta) &= \mathbb{P}_x^{\beta,\ell_\beta}\Big(\sum_{k=0}^{\lfloor \ell_\beta \rfloor}(\sigma_k - \sigma_{k-1}) \le \ell_\beta\Big)\\ &\le \mathbb{P}_x^{\beta,\ell_\beta}\Big(\sum_{k=0}^{\lfloor \ell_\beta/2 \rfloor -1}(\sigma_{2k+1} - \sigma_{2k}) \le \ell_\beta\Big)\\ &\le \mathbb{E}_x^{\beta,\ell_\beta} \exp\Big(\gamma\ell_\beta - \gamma \sum_{k=0}^{\lfloor \ell_\beta/2 \rfloor -1}(\sigma_{2k+1} - \sigma_{2k})\Big)\\ &= \exp\Big(\gamma\ell_\beta + (\lfloor \ell_\beta/2 \rfloor - 1)\log \mathbb{E}_x^{\beta,\ell_\beta} \mathrm{e}^{-\gamma(\sigma_1-\sigma_0)}\Big),\end{aligned}$$

where in the last equality we used that the random variables $\sigma_{2k+1} - \sigma_{2k}$, $k = 0, \dots, \lfloor \ell_\beta/2 \rfloor - 1$, are i.i.d. and independent of the initial condition x when X is sampled according to $\mathbb{P}_x^{\beta,\ell_\beta}$. More precisely, from the strong Markov property and the symmetry of the potential W, the law of $\sigma_{2k+1} - \sigma_{2k}$ is the same as the law of the hitting time τ of the point $-1 + \rho_2$ under $\mathbb{P}_{-1+\rho_1}^{\beta,\ell_\beta}$.

To estimate $\mathbb{E}_{-1+\rho_1}^{\beta,\ell_\beta}(\mathrm{e}^{-\gamma\tau})$ we use that $f(x) := \mathbb{E}_x^{\beta,\ell_\beta}(\mathrm{e}^{-\gamma\tau})$ can be characterized as the unique bounded solution to

$$\begin{cases} L_\beta f = \gamma f & \text{on } (-\infty, -1+\rho_2)\\ f(-1+\rho_2) = 1, \end{cases}$$

where $L_\beta f = \frac{1}{2\beta} f'' - U_\beta' f'$. Recalling (3.1), by ground state transformation, $h = f\psi_\beta$ satisfies

$$\begin{cases} H_\beta h = (E_{0,\beta} - \gamma)h & \text{on } (-\infty, -1+\rho_2)\\ h(-1+\rho_2) = \psi_\beta(-1+\rho_2). \end{cases}$$

Choosing $\gamma > \lim_\beta E_{0,\beta} = 1$, it follows that $h'' \ge 2\beta(\gamma - E_{0,\beta}) > 0$ for any β large enough. Since h is bounded, it is therefore a non decreasing function, whence $h(-1+\rho_1) \le h(-1+\rho_2) = \psi_\beta(-1+\rho_2)$ for any β large enough. We thus obtain $f(-1+\rho_1) \le \psi_\beta(-1+\rho_2)/\psi_\beta(-1+\rho_1)$. By Proposition 3.1 (iii) there exists $C_\rho > 0$ such that for any β large enough $f(-1+\rho_1) \le \mathrm{e}^{-\beta C_\rho}$. The statement follows. □

We next analyze the transition probabilities of the chain (Y_k) for $\beta \to \infty$. The transitions from $-1+\rho_2$ to $1-\rho_1$ and, symmetrically, $1-\rho_2$ to $-1+\rho_1$ correspond to atypical behavior of the underlying diffusion (2.15).

Lemma 5.4.

$$\varlimsup_{\rho_2\to 0}\varlimsup_{\rho_1\to 0}\varlimsup_{\beta\to\infty}\frac{1}{\beta}\log p_{\rho,\beta}\le -C_{\mathrm{W}}. \tag{5.3}$$

Proof. As follows by direct computation, the quasi-potential associated to the rate function in (4.8) is $2U$. In view of Proposition 4.4 and since $C_{\mathrm{W}}=2U(0)$, the proof is achieved by the arguments in [17, § 4.2]. □

Recalling $\mathcal{N}_{\rho,\beta}=\sup\{k\ge 0\colon \sigma_k<\ell_\beta\}$, we let $\mathcal{N}^{\pm}_{\rho,\beta}:=|\{1\le k\le \mathcal{N}_{\rho,\beta}\colon Y_{k-1}=\pm 1\mp\rho_2,\ Y_k=\mp 1\pm\rho_1\}|$. Otherwise stated, $\mathcal{N}^{+}_{\rho,\beta}$ (resp. $\mathcal{N}^{-}_{\rho,\beta}$) denotes the number of rare excursions from $1-\rho_2$ to $-1+\rho_1$ (resp. $-1+\rho_2$ to $1-\rho_1$).

Lemma 5.5. *For any $k\ge 0$,*

$$\varlimsup_{\rho_2\to 0}\varlimsup_{\rho_1\to 0}\varlimsup_{\beta\to\infty}\sup_{x\in\mathbb{R}}\frac{1}{\beta}\log\mathbb{P}^{\beta,\ell_\beta}_x(\mathcal{N}^{-}_{\rho,\beta}+\mathcal{N}^{+}_{\rho,\beta}\ge k)\le -k(C_{\mathrm{W}}-\alpha).$$

Proof. By the graphical construction of the discrete time Markov chain (Y_k), on the event $\{\mathcal{N}_{\rho,\beta}\le\ell_\beta\}$ the random variable $\mathcal{N}^{-}_{\rho,\beta}+\mathcal{N}^{+}_{\rho,\beta}$ is stochastically dominated by a random variable Z having binomial distribution with parameters $(2\lfloor\ell_\beta\rfloor,p)$. By the exponential Chebyshev inequality,

$$\mathbb{P}(Z\ge k)\le\exp\Big\{-2\lfloor\ell_\beta\rfloor I_p\Big(\frac{k}{2\lfloor\ell_\beta\rfloor}\Big)\Big\},\quad I_p(z)=z\log\frac{z}{p}+(1-z)\log\frac{1-z}{1-p}.$$

By elementary computations we conclude by Lemmata 5.3 and 5.4. □

In order to show exponential tightness, recalling the Fréchet-Kolmogorov compactness criterion, for $X\in L^p((0,1))$ we introduce

$$\omega_h(X)=\int_0^{1-h}\mathrm{d}s\,|X_{s+h}-X_s|^p+\int_0^h\mathrm{d}s\,|X_s|^p+\int_{1-h}^1\mathrm{d}s\,|X_s|^p,\quad h\in(0,1). \tag{5.4}$$

Lemma 5.6. *For each $L,\zeta>0$,*

$$\lim_{h\downarrow 0}\varlimsup_{\beta\to\infty}\sup_{|x|\le L}\frac{1}{\beta}\log(\mathbb{P}^{\beta,\ell_\beta}_x\circ\imath_\beta^{-1})(\omega_h>\zeta)=-\infty. \tag{5.5}$$

Proof. The last two terms in the right-hand side of (5.4) can be controlled by Lemma 4.1. By the same lemma, it suffices to control the first term in the right-hand side of (5.4) for $p=2$. Given $\delta>0$, define $I^{\pm}_\delta:=\{x\colon |x\mp 1|<\delta\}$ and introduce the sets

$$\begin{aligned}
D_{0,\rho}&=\{s\in(0,1-h)\colon ||X_s|-1|\ge\rho_2\}\cap\{s\in(0,1-h)\colon ||X_{s+h}|-1|\ge\rho_1\},\\
D_{\pm,\rho}&=\{s\in(0,1-h)\colon X_s\in I^{\pm}_{\rho_2}\}\cap\{s\in(0,1-h)\colon ||X_{s+h}|-1|\ge\rho_1\},\\
\hat{D}_{\pm,\rho}&=\{s\in(0,1-h)\colon X_{s+h}\in I^{\pm}_{\rho_1}\}\cap\{s\in(0,1-h)\colon ||X_s|-1|\ge\rho_2\},
\end{aligned}$$

$A_{\pm,\rho} = \{s \in (0, 1-h)\colon X_s \in I^{\pm}_{\rho_2}, X_{s+h} \in I^{\pm}_{\rho_1}\}$,
$C_{\pm,\rho} = \{s \in (0, 1-h)\colon X_s \in I^{\pm}_{\rho_2}, X_{s+h} \in I^{\mp}_{\rho_1}\}$,

that form a partition of $(0, 1-h)$. Letting

$$c_{0,\rho} = \sup\left\{\frac{|x|^2}{W(x)}\colon ||x|-1| \geq \rho_1\right\}, \quad c_{\pm,\rho} = \sup\left\{\frac{|x-(\pm 1)|^2}{W(x)}\colon ||x|-1| \geq \rho_1\right\},$$

we have

$$\int_{D_{0,\rho}} \mathrm{d}s\, |X_{s+h} - X_s|^2 \leq 4c_{0,\rho} \int_0^1 \mathrm{d}s\, W(X_s),$$

while, for $D_\rho = D_{\pm,\rho}, \hat{D}_{\pm,\rho}$,

$$\int_{D_\rho} \mathrm{d}s\, |X_{s+h} - X_s|^2 \leq 2\left(\rho_2^2 + c_{\pm,\rho} \int_0^1 \mathrm{d}s\, W(X_s)\right).$$

Furthermore,

$$\int_{A_{\pm,\rho}} \mathrm{d}s\, |X_{s+h} - X_s|^2 \leq (2\rho_2)^2.$$

Finally, recalling the stopping times σ_k introduced in (5.2) and the $\mathcal{N}^{\pm}_{\rho,\beta}$'s defined in Lemma 5.5, let $S^{\pm}$ be the ordered collection of σ_{2k+1} such that $Y_{2k+1} = \pm 1 \mp \rho_2$ and $Y_{2k+2} = \mp 1 \pm \rho_1$. Set also $S^{\pm} = \{\ell_\beta s^{\pm}_1, \ldots, \ell_\beta s^{\pm}_{\mathcal{N}^{\pm}_{\beta,\rho}}\}$. By construction,

$$C_{\pm,\rho} \subset \bigcup_{i=1}^{\mathcal{N}^{\pm}_{\rho,\beta}} [s^{\pm}_i - h, s^{\pm}_i],$$

so that

$$\int_{C_{\pm,\rho}} \mathrm{d}s\, |X_{s+h} - X_s|^2 \leq [2(1+\rho_2)]^2 \mathcal{N}^{\pm}_{\rho,\beta}\, h.$$

Gathering the previous estimates and using Lemmata 4.2 (i) and 5.5 the statement follows. □

Proof of Theorem 2.1: exponential tightness. We show that the family of probability measures $\{\mu_\beta\}$ on $L^p((0,1))$ is exponentially tight, that is, there exists a sequence of compacts $\mathcal{K}_j \subset L^p((0,1))$ such that

$$\lim_{j\to\infty} \varlimsup_{\beta\to\infty} \frac{1}{\beta} \log \mu_\beta(\mathcal{K}_j^{\mathrm{c}}) = -\infty.$$

Recalling (5.4) and the Fréchet-Kolmogorov compactness criterion, by a straightforward inclusion of events, see, for example, [6, § 8], it suffices to show that for each $\zeta > 0$

$$\lim_{h\downarrow 0} \varlimsup_{\beta} \frac{1}{\beta} \log \mu_\beta(\omega_h > \zeta) = -\infty.$$

This follows directly from Lemmata 5.1 and 5.6. □

Upper Bound

We first prove the following local upper bound.

Lemma 5.7. *Fix* $m \in \mathrm{BV}\big((0,1);\{-1,1\}\big)$ *and sequences* $m_k \to m$ *in* $L^p((0,1))$, $\zeta_k \to 0$. *Then*

$$\varlimsup_{k\to\infty}\varlimsup_{\beta\to\infty}\frac{1}{\beta}\log\mu_\beta(\mathcal{O}^p_{\zeta_k}(m_k)) \le -(C_{\mathrm{W}}-\alpha)|S(m)|.$$

Proof. Recalling assumption (2.8), by Lemma 5.2 it suffices to show the statement with μ_β replaced by $\mathbb{P}^{\beta,\ell_\beta}_{\pi_\beta}\circ\iota_\beta^{-1}$ and $\mathcal{O}^p_{\zeta_k}(m_k)$ replaced by

$$\Big\{X \in L^p((0,1))\colon \int_{T_\beta/\ell_\beta}^{1-T_\beta/\ell_\beta} \mathrm{d}s\,|X_s - m_k(s)|^p < (2\zeta_k)^p\Big\},$$

where T_β satisfies $\lim_\beta \beta^{-1}T_\beta = \infty$ and $\lim_\beta \ell_\beta^{-1}T_\beta = 0$. By Proposition 3.1 (iv) and Lemma 4.1, the previous statement is proven once we show that for each sequence $\zeta_k \to 0$ and $L > 0$

$$\varlimsup_{k\to\infty}\varlimsup_{\beta\to\infty}\sup_{|x|\le L}\frac{1}{\beta}\log\big(\mathbb{P}^{\beta,\ell_\beta}_x\circ\iota_\beta^{-1}\big)\big(\mathcal{O}^p_{\zeta_k}(m_k)\big) \le -(C_{\mathrm{W}}-\alpha)|S(m)|. \tag{5.6}$$

To prove it, we first note that there exists a sequence $\zeta'_k \to 0$ such that the inclusion $\mathcal{O}_{\zeta_k}(m_k) \subset \mathcal{O}_{\zeta'_k}(m)$ holds for every k. Moreover, recalling the definition of $\mathcal{N}^{\pm}_{\rho,\beta}$ in Lemma 5.5, there exist sequences $0 < \rho_{1,k} < \rho_{2,k}$ with $\rho_{2,k} \to 0$ such that

$$\mathcal{O}_{\zeta'_k}(m) \subset \iota_\beta^{-1}\Big(\mathcal{N}^+_{\rho_k,\beta} + \mathcal{N}^-_{\rho_k,\beta} \ge |S(m)|\Big), \quad k \in \mathbb{N},$$

where $\rho_k = (\rho_{1,k}, \rho_{2,k})$. Therefore (5.6) follows from Lemma 5.5. □

Proof of Theorem 2.1: upper bound on compacts. The proof is achieved by showing that the local upper bound in Lemma 5.7 implies the upper bound for compacts. The corresponding somewhat technical but general argument is based on a min-max lemma and detailed for completeness. Let $\mathrm{BV}_{\le n} = \{m \in \mathrm{BV}\big((0,1);\{-1,1\}\big) : |S(m)| \le n\}$ so that $\mathrm{BV}\big((0,1);\{-1,1\}\big) = \bigcup_n \mathrm{BV}_{\le n}$. Since $\mathrm{BV}_{\le n}$ is a compact subset of $L^p((0,1))$, for each $\zeta > 0$ there exists $K \in \mathbb{N}$ and $m_1,\dots,m_K$ in $\mathrm{BV}_{\le n}$ such that $\mathrm{BV}_{\le n} \subset \bigcup_i \mathcal{O}^p_\zeta(m_i) =: \mathcal{A}_{n,\zeta}$.

For $\zeta > 0$ and $m \in \mathrm{BV}_{\le n}$ set

$$j_\zeta(m) = -\varlimsup_\beta \frac{1}{\beta}\log\mu_\beta(\mathcal{O}^p_\zeta(m)).$$

Let also the function $J^1_{n,\zeta} : \mathcal{A}_{n,\zeta} \to [0,\infty)$ be defined by

$$J^1_{n,\zeta}(X) = \min_{i : X \in \mathcal{O}^p_\zeta(m_i)} j_\zeta(m_i).$$

Let $\{\sigma_k\}$ be as defined in (5.2) and $\mathcal{N}^{\pm}_{\rho,\beta}$ as in Lemma 5.5. Introduce the event $\mathcal{B}_n = \iota_\beta^{-1}\Big(\mathcal{N}^+_{\beta,\rho} + \mathcal{N}^-_{\beta,\rho} \ge n\Big)$ and set

$$c_n = -\varlimsup_\beta \frac{1}{\beta} \log \mu_\beta(\mathcal{B}_n) = -\varlimsup_\beta \frac{1}{\beta} \log \mu_{\beta,\ell_\beta}(\mathcal{N}^+_{\beta,\rho} + \mathcal{N}^-_{\beta,\rho} \ge n).$$

By the Urysohn lemma there exists a continuous function $J^2_{n,\zeta} \colon L^p((0,1)) \to [0,\infty)$ such that

$$J^2_{n,\zeta}(X) = \begin{cases} J^1_{n,\zeta}(X) & \text{if } X \in \mathcal{A}_{n,\zeta}, \\ c_n & \text{if } X \in \mathcal{A}^{\mathrm{c}}_{n,2\zeta}. \end{cases}$$

Set finally $J_{n,\zeta} = J^2_{n,\zeta} \wedge c_n$.

We claim that for each ζ, n and each open $\mathcal{G} \subset L^p((0,1))$

$$\varlimsup_\beta \frac{1}{\beta} \log \mu_\beta(\mathcal{G}) \le -\inf_{X\in\mathcal{G}} J_{n,\zeta}(X) \tag{5.7}$$

and that, for each $X \in L^p((0,1))$,

$$\varliminf_n \varliminf_{\zeta\to 0} J_{n,\zeta}(X) \ge I(X), \tag{5.8}$$

where I is the rate function defined in (2.10). Since $X \mapsto J_{n,\zeta}(X)$ is continuous, by the min-max lemma in [26, App. 2, Lemma 3.2], this claim implies the upper bound on compacts.

In order to prove (5.7), we introduce the sets

$$\mathcal{W}_\delta = \Big\{X \in L^p((0,1)) \colon \int_0^1 \mathrm{d}s\, W(X_s)) \le \delta\Big\}, \tag{5.9}$$

$$\mathcal{X}_R = \Big\{X \in L^p((0,1)) \colon \sup_{0<t<\ell_\beta} |X_t| \le R\Big\}, \tag{5.10}$$

and observe that, by Lemmata 4.1, 4.2 (i), and 5.1, for each $\delta, R > 0$,

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \mu_\beta(\mathcal{W}^{\mathrm{c}}_\delta) = -\infty, \quad \lim_{\beta\to\infty} \frac{1}{\beta} \log \mu_\beta(\mathcal{X}^{\mathrm{c}}_R) = -\infty. \tag{5.11}$$

We then write

$$\mu_\beta(\mathcal{G}) \le \mu_\beta(\mathcal{G} \cap \mathcal{B}_n \cap \mathcal{W}_\delta \cap \mathcal{X}_R) + \mu_\beta(\mathcal{W}^{\mathrm{c}}_\delta) + \mu_\beta(\mathcal{X}^{\mathrm{c}}_R) + \mu_\beta(\mathcal{B}^{\mathrm{c}}_n).$$

By (5.11) and the above definitions of j_ζ and c_n, to complete the proof of (5.7) it suffices to show that there exist $\delta = \delta(n,\zeta,R) > 0$ such that $\mathcal{B}_n \cap \mathcal{W}_\delta \cap \mathcal{X}_R \subset \mathcal{A}_{n,\zeta}$. Given $X \in C([0,1])$, satisfying $\mathcal{N}^+_{\rho,\beta}(\iota_\beta^{-1}X) + \mathcal{N}^+_{\rho,\beta}(\iota_\beta^{-1}X) = k \le n$, we recursively define $(s_0,\alpha_0),\dots,(s_k,\alpha_k)$ as follows. Let $s_0 := \inf\{s \in [0,1] \colon |X_s| \ge 1-\rho_1\}$ and set $\alpha_0 = +$ if $X_{s_0} \ge 1-\rho_1$, $\alpha_0 = -$ if $X_{s_0} \le -1+\rho_1$. Define iteratively $\alpha_j = -\alpha_{j-1}$, $s_j = \inf\{s \in [s_{j-1},1] \colon X_s = 1-\rho_1\}$ if $\alpha_{j-1} = +$ and

$s_j = \inf\{s \in [s_{j-1}, 1] \colon X_s = -1 + \rho_1\}$ if $\alpha_{j-1} = -$. Let now $C = C(\rho_1) < +\infty$ be such that

$$|x+1|^2 \le C W(x) \text{ for } x \le 1 - \rho_1 \quad \text{and} \quad |x-1|^2 \le C W(x) \text{ for } x \ge -1 + \rho_1,$$

that exists since W has quadratic minima at $x = \pm 1$. The previous bound implies that if $X \in \mathcal{W}_\delta \cap \mathcal{X}_R$ with δ small enough then $X \in \mathcal{O}^p_\zeta(m^{\alpha_0,k}_{s_1,\ldots,s_k})$.

To prove (5.8) we first observe that, by Lemmata 5.5 and 5.1, $c_n \to \infty$. Moreover Lemma 5.7 implies that $\underline{\lim}_{\zeta \to 0} j_\zeta(m) = I(m)$ uniformly for m in compacts. Since $\bigcup_n \bigcap_\zeta \mathcal{A}_{n,\zeta} = \mathrm{BV}((0,1); \{-1,1\})$ the claim follows. □

Lower Bound

We start by proving the large deviation lower bound for the law of the diffusion (2.15). Recall that $m^{\pm,0} := \pm 1$ and the definition of $m^{\pm,n}_{s_1,\ldots,s_n}$ given in (2.12).

Lemma 5.8. *Fix $n \ge 0$, and, if $n \ge 1$, also $0 < s_1 < \cdots < s_n < 1$. For any $\delta \in (0,1)$ and any open L^p-neighborhood $\mathcal{O}$ of $m^{\pm,n}_{s_1,\ldots,s_n}$,*

$$\varliminf_{\beta\to\infty} \inf_{x \in I^\pm_\delta} \frac{1}{\beta} \log(\mathbb{P}^{\beta,\ell_\beta}_x \circ \iota_\beta^{-1})(\mathcal{O}) \ge -(C_{\mathrm{W}} - \alpha) n,$$

where $I^\pm_\delta := (\pm 1 - \delta, \pm 1 + \delta)$.

Proof. For the ease of presentation we detail first the cases $n = 0, 1$. For the case $n = 0$, it is enough to show that for each $\eta > 0$,

$$\lim_{\beta\to\infty} \mathbb{P}^{\beta,\ell_\beta}_x \Big(\frac{1}{\ell_\beta} \int_0^{\ell_\beta} \mathrm{d}t\, |X_t \mp 1|^p > \eta \Big) = 0 \quad \text{uniformly for } x \in I^\pm_\delta. \tag{5.12}$$

Recalling (2.8), pick $x_0 \in (0, 1-\delta)$ such that $2U(x_0) > \alpha$ and let $\tau^\pm = \inf\{t \ge 0 \colon X_t = \pm x_0\}$ be the hitting time of $\pm x_0$. By Proposition 4.4 and the argument in [17, Chap. 4, Theorem 4.2] it follows that

$$\lim_{\beta\to\infty} \mathbb{P}^{\beta,\ell_\beta}_x (\tau^\pm < \ell_\beta) = 0 \quad \text{uniformly for } x \in I^\pm_\delta. \tag{5.13}$$

Thus (5.12) follows from Lemma 4.2 (i).

For the case $n = 1$, by symmetry, it is enough to consider $m^-_{s_1}$. Set $t_1 = \ell_\beta s_1$ and $\widehat{X}_t = X_{(0\vee t)\wedge\ell_\beta}$. For $T, \eta > 0$, recalling that θ denotes the time shift, we define

$$\begin{aligned} \mathcal{A}^- &= \Big\{ X \colon \widehat{X}_{t_1 - T} \in I^-_\delta \text{ and } \frac{1}{\ell_\beta} \int_0^{t_1 - T} \mathrm{d}t\, |\widehat{X}_t + 1|^p < \eta \Big\}, \\ \mathcal{A}^+ &= \Big\{ X \colon \widehat{X}_{t_1 + T} \in I^+_\delta \text{ and } \frac{1}{\ell_\beta} \int_{t_1 + T}^{\ell_\beta} \mathrm{d}t\, |\widehat{X}_t - 1|^p < \eta \Big\}, \\ \mathcal{A}^0 &= \Big\{ X \colon \widehat{X}_{t_1 \pm T} \in I^\pm_\delta \text{ and } |\widehat{X}_t| < 1 + \delta \text{ if } |t - t_1| < T \Big\}. \end{aligned}$$

We define, for $a \in \{0, -, +\}$ and $k \in \mathbb{Z}$, $\mathcal{A}_k^a = \theta_{2kT}\mathcal{A}^a$. It is straightforward to verify that, given $\mathcal{O}$ as in the statement, there exist $\eta, \zeta > 0$ such that, for β large enough and $\bar{k} = \lfloor \ell_\beta \zeta / T \rfloor$,

$$\iota_\beta^{-1}(\mathcal{O}) \supset \bigcup_{k=-\bar{k}}^{\bar{k}} \mathcal{C}_k, \quad \mathcal{C}_k = \mathcal{A}_k^- \cap \mathcal{A}_k^0 \cap \mathcal{A}_k^+.$$

By the Bonferroni inequality,

$$\begin{aligned}(\mathbb{P}_x^{\beta,\ell_\beta} \circ \iota_\beta^{-1})(\mathcal{O}) \geq \mathbb{P}_x^{\beta,\ell_\beta}\Big(\bigcup_{k=-\bar{k}}^{\bar{k}} \mathcal{C}_k\Big) &\geq \sum_{k=-\bar{k}}^{\bar{k}} \mathbb{P}_x^{\beta,\ell_\beta}(\mathcal{C}_k) \\ &\quad - \sum_{-\bar{k} \leq h < k \leq \bar{k}} \mathbb{P}_x^{\beta,\ell_\beta}(\mathcal{C}_h \cap \mathcal{C}_k).\end{aligned}$$

We claim that

$$\varliminf_{T\to\infty} \varliminf_{\beta\to\infty} \inf_{|k|\leq\bar{k}} \inf_{x\in I_\delta^-} \frac{1}{\beta} \log \mathbb{P}_x^{\beta,\ell_\beta}(\mathcal{C}_k) \geq -C_{\mathrm{W}}, \tag{5.14}$$

and

$$\varlimsup_{T\to\infty} \varlimsup_{\beta\to\infty} \sup_{-\bar{k}\leq h<k\leq\bar{k}} \sup_{x\in I_\delta^-} \frac{1}{\beta} \log \mathbb{P}_x^{\beta,\ell_\beta}(\mathcal{C}_h \cap \mathcal{C}_k) \leq -2C_{\mathrm{W}}, \tag{5.15}$$

which, recalling (2.8), readily imply the statement for $n = 1$.

In order to prove (5.14) we first observe that (5.12) implies

$$\lim_{\beta\to\infty} \mathbb{P}_x^{\beta,\ell_\beta}(\theta_{t_1+T}\mathcal{A}^+) = 1 \quad \text{uniformly for } x \in I_\delta^+.$$

Moreover, as follows from Lemma 4.3 and again (5.12),

$$\lim_{\beta\to\infty} \mathbb{P}_x^{\beta,\ell_\beta}(\mathcal{A}_k^-) = 1 \quad \text{uniformly for } x \in I_\delta^- \text{ and } |k| \leq \bar{k}.$$

Finally, from Proposition 4.4 and $C_{\mathrm{W}} = 2U(0)$, we deduce

$$\varliminf_{T\to\infty} \varliminf_{\beta\to\infty} \inf_{y\in I_\delta^-} \frac{1}{\beta} \log P_x^{\beta,\ell_\beta}(\theta_{t_1-T}\mathcal{A}^0) \geq -C_{\mathrm{W}}. \tag{5.16}$$

By the Markov property and using the identities

$$\theta_{t_1+(2k+1)T}\mathbf{1}_{\mathcal{A}_k^+} = \mathbf{1}_{\theta_{t_1+T}\mathcal{A}^+}, \qquad \theta_{t_1+(2k-1)T}\mathbf{1}_{\mathcal{A}_k^0} = \mathbf{1}_{\theta_{t_1-T}\mathcal{A}^0},$$

we have

$$\begin{aligned}\mathbb{P}_x^{\beta,\ell_\beta}(\mathcal{C}_k) = \mathbb{P}_x^{\beta,\ell_\beta}(\mathcal{A}_k^- \cap \mathcal{A}_k^0 \cap \mathcal{A}_k^+) &= \mathbb{E}_x^{\beta,\ell_\beta}(\mathbf{1}_{\mathcal{A}_k^-}\mathbf{1}_{\mathcal{A}_k^0}\,\mathbb{E}^{\beta,\ell_\beta}_{X_{t_1+(2k+1)T}}(\mathbf{1}_{\theta_{t_1+T}\mathcal{A}^+})) \\ &\geq \inf_{y\in I_\delta^+} \mathbb{P}_y^{\beta,\ell_\beta}(\theta_{t_1+T}\mathcal{A}^+)\,\mathbb{E}_x^{\beta,\ell_\beta}(\mathbf{1}_{\mathcal{A}_k^-}\mathbf{1}_{\mathcal{A}_k^0}) \\ &= \inf_{y\in I_\delta^+} \mathbb{P}_y^{\beta,\ell_\beta}(\theta_{t_1+T}\mathcal{A}^+)\,\mathbb{E}_x^{\beta,\ell_\beta}(\mathbf{1}_{\mathcal{A}_k^-}\,\mathbb{E}^{\beta,\ell_\beta}_{X_{t_1+(2k-1)T}}(\mathbf{1}_{\theta_{t_1-T}\mathcal{A}^0})) \\ &\geq \inf_{y\in I_\delta^+} \mathbb{P}_y^{\beta,\ell_\beta}(\theta_{t_1+T}\mathcal{A}^+) \inf_{z\in I_\delta^-} \mathbb{P}_z^{\beta,\ell_\beta}(\theta_{t_1-T}\mathcal{A}^0)\,\mathbb{P}_x^{\beta}(A_k^-),\end{aligned}$$

and the bound (5.14) follows.

To prove (5.15), again by Proposition 4.4 and $C_{\mathrm{W}} = 2U(0)$, we deduce

$$\varlimsup_{T\to\infty} \varlimsup_{\beta\to\infty} \sup_{y\in I_\delta^-} \frac{1}{\beta} \log \mathbb{P}_x^{\beta,\ell_\beta}(\theta_{t_1-T}\mathcal{A}^0) \le -C_{\mathrm{W}}. \tag{5.17}$$

By the Markov property, for $h < k$,

$$\begin{aligned}
\mathbb{P}_x^{\beta,\ell_\beta}(\mathcal{C}_h \cap \mathcal{C}_k) &\le \mathbb{P}_x^{\beta,\ell_\beta}(\mathcal{A}_h^0 \cap \mathcal{A}_k^0) = \mathbb{E}_x^{\beta,\ell_\beta}(\mathbf{1}_{\mathcal{A}_h^0}\, \mathbb{E}^{\beta,\ell_\beta}_{X_{t_1+(2k-1)T}}(\mathbf{1}_{\theta_{t_1-T}\mathcal{A}^0}))\\
&\le \sup_{y\in I_\delta^-} \mathbb{P}_y^{\beta,\ell_\beta}(\mathbf{1}_{\theta_{t_1-T}\mathcal{A}^0})\, \mathbb{E}_x^{\beta,\ell_\beta}(\mathbb{E}^{\beta,\ell_\beta}_{X_{t_1+(2h-1)T}}(\mathbf{1}_{\theta_{t_1-T}\mathcal{A}^0}))\\
&\le \sup_{y\in I_\delta^-} \big(\mathbb{P}_y^{\beta,\ell_\beta}(\mathbf{1}_{\theta_{t_1-T}\mathcal{A}^0})\big)^2,
\end{aligned}$$

and the bound (5.15) follows by (5.17).

For the general case, by symmetry, it is enough to consider $m^{-,n}_{s_1,\dots,s_n}$. Set $t_i = \ell_\beta s_i$, $i = 1,\dots,n$, and for $\eta, T > 0$ introduce the events $\mathcal{A}^-$ as before and

$$\begin{aligned}
\mathcal{A}^+ &= \Big\{X\colon \widehat{X}_{t_n+T} \in I_\delta^+ \text{ and } \frac{1}{\ell_\beta}\int_{t_n+T}^{\ell_\beta} \mathrm{d}t\,|\widehat{X}_t - 1|^p < \eta\Big\},\\
\mathcal{A}^{0,i} &= \Big\{X\colon \widehat{X}_{t_i\pm T} \in I_\delta^\pm \text{ and } |\widehat{X}_t| < 1+\delta \text{ if } |t-t_i| < T\Big\}.
\end{aligned}$$

As before, for $a \in \{\{0,i\}, -, +\}$ and $k \in \mathbb{Z}$ set $\mathcal{A}_k^a = \theta_{2kT}\mathcal{A}^a$, and for $k, h \in \mathbb{Z}$, we let

$$\mathcal{B}_{k,h}^{i,i+1} = \Big\{X\colon X_{t^+_{i,k}}, X_{t^-_{i+1,h}} \in I_\delta^i, \text{ and } \frac{1}{\ell_\beta}\int_{t^+_{i,k}}^{t^-_{i+1,h}} \mathrm{d}t\,|X_t - (-1)^{i+1}|^p < \eta\Big\},$$

where $t^\pm_{i,k} = t_i + (2k \pm 1)T$ and $I_\delta^i = I_\delta^+$, respectively $I_\delta^i = I_\delta^-$, if i is odd, respectively even. Again, it is straightforward to verify that, given $\mathcal{O}$ as in the statement, there exist $\eta, \zeta > 0$ such that, for β large enough and $\bar{k} = \lfloor \ell_\beta\zeta/T\rfloor$,

$$\iota_\beta^{-1}(\mathcal{O}) \supset \bigcup_{k_1=-\bar{k}}^{\bar{k}} \cdots \bigcup_{k_n=-\bar{k}}^{\bar{k}} \mathcal{C}_{k_1,\dots,k_n},$$

with

$$\mathcal{C}_{k_1,\dots,k_n} = \mathcal{A}^-_{k_1} \cap \mathcal{A}^{0,1}_{k_1} \cap \mathcal{B}^{1,2}_{k_1,k_2} \cap \mathcal{A}^{0,2}_{k_2} \cap \cdots \cap \mathcal{A}^{0,n-1}_{k_{n-1}} \cap \mathcal{B}^{n-1,n}_{k_{n-1},k_n} \cap \mathcal{A}^{0,n}_{k_n} \cap \mathcal{A}^+_{k_n}.$$

From (5.12), (5.16) and (5.13), by using the Markov property as in the case $n = 1$, we get

$$\varliminf_{T\to\infty} \varliminf_{\beta\to\infty} \frac{1}{\beta} \log \mathbb{P}_x^{\beta,\ell_\beta}(\mathcal{C}_{k_1,\dots,k_n}) \ge -nC_{\mathrm{W}},$$

uniformly for $|k_i| \leq \bar{k}$ and $x \in I_\delta^-$. Again by the Markov property and by (5.17),

$$\varlimsup_{T\to\infty} \varlimsup_{\beta\to\infty} \frac{1}{\beta} \log \mathbb{P}_x^{\beta,\ell_\beta}(\mathcal{C}_{h_1,\dots,h_n} \cap \mathcal{C}_{k_1,\dots,k_n}) \leq -(n + |\{i : k_i \neq h_i\}|)C_{\mathrm{W}},$$

uniformly for $|h_i|, |k_i| \leq \bar{k}$, $(h_1,\dots,h_n) \neq (k_1,\dots,k_n)$ and $x \in I_\delta^-$. By using the Bonferroni inequality as before the statement follows. □

Proof of Theorem 2.1: lower bound. Fix $n \geq 0$ and, if $n \geq 1$, also $0 < s_1 < \cdots < s_n < 1$. It is enough to show that for any open L^p-neighborhood $\mathcal{O}$ of $m^{\pm,n}_{s_1,\dots,s_n}$,

$$\varliminf_{\beta\to\infty} \frac{1}{\beta} \log \mu_\beta(\mathcal{O}) \geq -(C_{\mathrm{W}} - \alpha)n.$$

By symmetry, it is enough to consider $m^{-,n}_{s_1,\dots,s_n}$. By the representation (4.1) and (5.1),

$$\begin{aligned}
\mu_\beta(\mathcal{O}) &= \int_{\mathbb{R}^2} \wp_\beta(\mathrm{d}x\,\mathrm{d}y)(\mathbb{P}_{x,y}^{\beta,\ell_\beta} \circ \iota_\beta^{-1})(\mathcal{O}) \\
&= \frac{1}{Z_\beta} \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} \mathbb{P}_x^{\beta,\ell_\beta}(X_{\ell_\beta} \in \mathrm{d}y) \frac{\psi_{0,\beta}(x)}{\psi_{0,\beta}(y)} (\mathbb{P}_{x,y}^{\beta,\ell_\beta} \circ \iota_\beta^{-1})(\mathcal{O}) \\
&\geq \frac{1}{Z_\beta} \inf_{y\in\mathbb{R}} \frac{1}{\psi_{0,\beta}(y)} \int_{\mathbb{R}} \mathrm{d}x\, \psi_{0,\beta}(x) (\mathbb{P}_x^{\beta,\ell_\beta} \circ \iota_\beta^{-1})(\mathcal{O}) \\
&\geq \frac{1}{Z_\beta} \inf_{y\in\mathbb{R}} \frac{1}{\psi_{0,\beta}(y)} \inf_{z\in I_\delta^-} (\mathbb{P}_z^{\beta,\ell_\beta} \circ \iota_\beta^{-1})(\mathcal{O}) \int_{I_\delta^-} \mathrm{d}x\, \psi_{0,\beta}(x).
\end{aligned}$$

By items (iii) and (iv) in Proposition 3.1,

$$\varliminf_{\beta\to\infty} \frac{1}{\beta} \inf_{y\in\mathbb{R}} \log \frac{1}{\psi_{0,\beta}(y)} \geq 0,$$

and the proof is concluded by Corollary 3.3, (3.16), and Lemma 5.8. □

6. Sharp Asymptotics

The aim of this section is to prove the sharp asymptotics stated in Theorem 2.2. Accordingly, we fix $p \in [1,\infty)$, $n \geq 0$, and the sequences $(\rho_{k,\beta})_{\beta>0}$, $k = 1, 2$, meeting the conditions in the statement.

We choose a sequence $(T_\beta)_{\beta>0}$ such that

$$\lim_{\beta\to\infty} \frac{T_\beta}{\beta^{3/2}} = \infty, \quad \lim_{\beta\to\infty} \frac{T_\beta}{\ell_\beta} = 0, \quad \lim_{\beta\to\infty} \frac{T_\beta}{\ell_\beta \rho_{1,\beta}^p} = \infty, \tag{6.1}$$

which is possible in view of assumption (2.11). We recursively define the sequence of stopping times (τ_k) on $C([0,\ell_\beta])$, by setting $\tau_0 = 0$ and, for $k \in \mathbb{N}$,

$$\tau_k = \inf\{t \in [T_\beta + \tau_{k-1}, \ell_\beta - 2T_\beta) : X_t = 0\} \wedge (\ell_\beta - 2T_\beta),$$

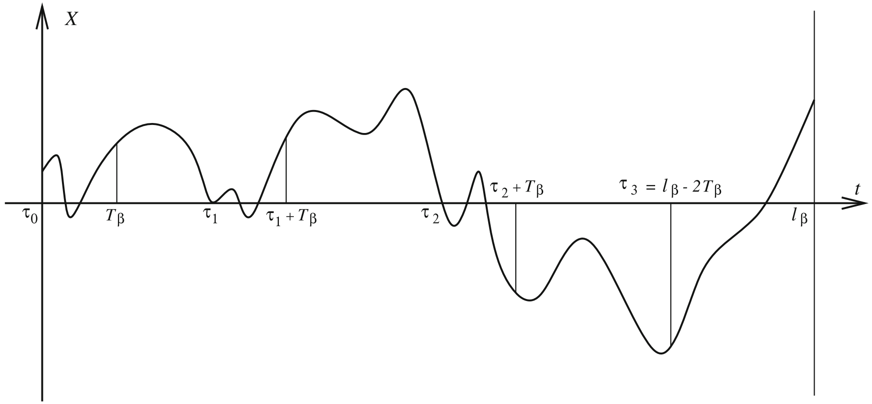

Fig. 2. An example of stopping times τ_k, with $\mathcal{Z} = 2$ and $\mathcal{N} = 1$

that is, (τ_k) is the sequence of passages at zero, recursively defined by requiring that they are sufficiently far apart from each other, see Figure 2. Further, we denote by

$$\mathcal{Z} = \left|\{k \in \mathbb{N} : \tau_k < \ell_\beta - 2T_\beta\}\right|,$$

the number of passages occurring before ℓ_β. Observe that $\tau_k - \tau_{k-1} \geq T_\beta$ for $k \leq \mathcal{Z}$. Let also

$$\mathcal{N} = \left|\{k \in \mathbb{N} : X_{T_\beta+\tau_{k-1}} X_{T_\beta+\tau_k} < 0\}\right|$$

be the number of relevant sign changes of the process taking into account the "padding time" T_β, see again Figure 2. We finally define

$$\mathcal{B}_{n,\delta} = \bigcap_{k=1}^{n} \{\tau_k - \tau_{k-1} > \ell_\beta \delta\} \cap \{\tau_n < \ell_\beta(1-\delta)\}.$$

Recall that for given strictly positive sequences $a_n, b_n, n \in \mathbb{N}$, we use the standard asymptotic notation $a_n \sim b_n$ when $a_n = b_n(1 + r_n)$, with $\lim_n r_n = 0$.

Proposition 6.1. *As* $\beta \to \infty$,

(i) $\mu_{\beta,\ell_\beta}(\mathcal{Z} \geq n) \sim \dfrac{2^n}{n!}\Big(\dfrac{\ell_\beta}{\bar{\ell}_\beta}\Big)^n$.

(ii) $\mu_{\beta,\ell_\beta}(\mathcal{N} \geq n) \sim \dfrac{1}{n!}\Big(\dfrac{\ell_\beta}{\bar{\ell}_\beta}\Big)^n$.

(iii) Let $(\delta_\beta)_{\beta>0}$ *be a sequence such that* $\lim_\beta \delta_\beta \ell_\beta / T_\beta = 0$. *Then*

$$\mu_{\beta,\ell_\beta}(\mathcal{N} = n,\ \mathcal{B}_{n,\delta_\beta}) \sim \frac{1}{n!}\Big(\frac{\ell_\beta}{\bar{\ell}_\beta}\Big)^n.$$

Postponing the proof of this proposition, we first conclude the proof of Theorem 2.2. To this end, we introduce the events

$$\mathcal{W}_\gamma = \left\{X : \frac{1}{\ell_\beta}\int_0^{\ell_\beta} \mathrm{d}t\, W(X_t) \leq \frac{1+\gamma}{\beta}\right\}, \qquad \mathcal{X}_R = \left\{X : \sup_{0<t<\ell_\beta} |X_t| \leq R\right\}.$$

Descriptively, these events are defined as sublevel sets of the integral mean of $W \circ X$ and the supremum norm of X, respectively.

Lemma 6.2. *Fix* $\gamma, R > 0$, $z \in \mathbb{N}$ *and* $i = 1, 2$. *Then, eventually as* $\beta \to \infty$,

$$\left[\imath_\beta \mathcal{O}_{\rho_{i,\beta}}(\mathcal{M}_{n-1})\right]^{\mathrm{c}} \cap \mathcal{W}_\gamma \cap \mathcal{X}_R \cap \{\mathcal{Z} \leq z\} \subset \{\mathcal{N} \geq n\}.$$

Proof. Let $X \in \left[\imath_\beta \mathcal{O}_{\rho_{i,\beta}}(\mathcal{M}_0)\right]^{\mathrm{c}} \cap \mathcal{W}_\gamma \cap \mathcal{X}_R \cap \{\mathcal{Z} \leq z\}$. Set

$$J = \{j \in \{1, \ldots, \mathcal{Z}(X)\} \colon X_{T_\beta+\tau_{j-1}} X_{T_\beta+\tau_j} < 0\}, \quad \bar{n} = |J|,$$

and, in the case $J \neq \emptyset$, let $j_1, \ldots, j_{\bar n}$ be an increasing enumeration of the elements of J. Assuming with no loss of generality that $X_{\tau_{j_1}+T_\beta} > 0$, let $m_J^- = m_{s_1,\ldots,s_{\bar n}}^{-,\bar n}$, with $s_k = \tau_{j_k}/\ell_\beta$, $k = 1, \ldots, \bar n$, if $\bar n \geq 1$ and $m_\emptyset^- = -1$. Since $X \in \mathcal{X}_R$,

$$\|\iota_\beta X - m_J^-\|_{L^p}^p \leq \frac{(3 + \mathcal{Z}(X))T_\beta}{\ell_\beta}(1 + R)^p + B_1(X) + B_2(X), \tag{6.2}$$

where

$$B_1(X) = \frac{1}{\ell_\beta} \sum_{k=1}^{\bar n+1} \sum_{m=0}^{j_k - j_{k-1}-1} \int_{T_\beta+\tau_{j_{k-1}+m}}^{\tau_{j_{k-1}+1+m}} \mathrm{d}t\, |X_t - (-1)^k|^p,$$

$$B_2(X) = \frac{1}{\ell_\beta} \int_{T_\beta+\tau_{\mathcal{Z}(X)}}^{\ell_\beta - 2T_\beta} \mathrm{d}t\, |X_t - (-1)^{\bar n+1}|^p\, \mathbf{1}_{\tau_{\mathcal{Z}(X)}+T_\beta<\ell_\beta-2T_\beta}.$$

From the assumptions on T_β and $\mathcal{Z}(X) \leq z$ it follows that the first term on the right-hand side of (6.2) satisfies $\lim_\beta \beta \frac{(3+\mathcal{Z}(X))T_\beta}{\ell_\beta}(1+R)^p = 0$. To estimate $B_1(X)$ we consider first the case $p \in [2, \infty)$. We observe that $X \geq 0$ (resp. $X \leq 0$) on $[T_\beta + \tau_{j_{k-1}+m}, \tau_{j_{k-1}+1+m}]$ for any k even (resp. k odd) and any m. Therefore, by using that $(x + 1)^2 \leq CW(x)$ for $x < 0$ and $(x - 1)^2 \leq CW(x)$ for $x > 0$, we deduce

$$B_1(X) \leq C(1 + R)^{p-2} \frac{1 + \gamma}{\beta}.$$

The term $B_2(X)$ satisfies the same bound, as can be shown with the same reasoning using that $X \geq 0$ (resp. $X \leq 0$) on $[T_\beta + \tau_{\mathcal{Z}}, \ell_\beta - 2T_\beta]$ if $\bar n$ is odd (resp. even). Hence, eventually as $\beta \to \infty$,

$$\|\iota_\beta X - m_J^-\|_{L^p} \leq C\beta^{-1/p}. \tag{6.3}$$

On the other hand, if $\bar n < n$, then $m_J^- \in \mathcal{O}_{\rho_\beta}(\mathcal{M}_{n-1})$ and thus $\|\iota_\beta X - m_J^-\|_{L^p} \geq \rho_{i,\beta}$, which, together with (6.3), contradicts the assumption on $\rho_{i,\beta}$. Therefore $\bar n \geq n$, which concludes the proof in the case $p \in [2, \infty)$. For $p \in [1, 2)$ we use $\|\iota_\beta X - m_J^-\|_{L^p} \leq \|\iota_\beta X - m_J^-\|_{L^2} \leq C\beta^{-1/2}$, where we used (6.3) with $p = 2$ in the second inequality. □

Lemma 6.3. *Fix* $\gamma, R > 0$ *and* $z \in \mathbb{N}$. *Let also* $(\delta_\beta)_{\beta>0}$ *be a positive sequence such that* $\lim_\beta \delta_\beta = 0$ *and* $\lim_\beta \delta_\beta \rho_{2,\beta}^{-p} = \infty$. *Then, eventually as* $\beta \to \infty$,

$$\{\mathcal{N} = n\} \cap \mathcal{B}_{n,\delta_\beta} \cap \mathcal{W}_\gamma \cap \mathcal{X}_R \cap \{\mathcal{Z} \leq z\} \subset \left[\iota_\beta \mathcal{O}_{\rho_{2,\beta}}(\mathcal{M}_{n-1})\right]^{\mathrm{c}}.$$

Proof. The statement is a direct consequence of the following two steps.
Step 1. If $X \in \{\mathcal{N} = n\} \cap \mathcal{B}_{n,\delta_\beta} \cap \mathcal{W}_\gamma \cap \mathcal{X}_R \cap \{\mathcal{Z} \leq z\}$ then there are $0 = s_0 < s_1 < \cdots < s_n < 1$ satisfying $s_k - s_{k-1} \geq \delta_\beta$, $k = 1, \ldots, n$, $s_n \leq 1 - \delta_\beta$ and $* \in \{-, +\}$ such that

$$\|\iota_\beta X - m_{s_1,\ldots,s_n}^{*,n}\|_{L^p} < \rho_{2,\beta}.$$

The proof of this step is achieved by arguing as in the proof of Lemma 6.2.

Step 2. Let $0 = s_0 < s_1 < \cdots < s_n < 1$ be such that $s_k - s_{k-1} \geq \delta_\beta, k = 1, \ldots, n$, $s_n \leq 1 - \delta_\beta$ and $* \in \{-, +\}$. Then

$$\operatorname{dist}_{L^p}(m^{*,n}_{s_1,\ldots,s_n}, \mathcal{M}_{n-1}) \geq 2(\delta_\beta/2)^{1/p}.$$

Consider the intervals $I_k = (s_k - \delta_\beta/2, s_k + \delta_\beta/2)$, $k = 1, \ldots, n$, which are pairwise disjoint. For any $m \in \mathcal{M}_{n-1}$ there exist an interval $I_{\bar{k}}$ on which m is constant. Hence,

$$\operatorname{dist}_{L^p}(m^{*,n}_{s_1,\ldots,s_n}, \mathcal{M}_{n-1})^p \geq 2^p \delta_\beta/2.$$

□

Proof of Theorem 2.2. For each $\gamma, R, z > 0$ and $k = 1, 2$, by Lemma 6.2, eventually as $\beta \to \infty$, we have

$$\mu_\beta([\mathcal{O}_{\rho_{k,\beta}}(\mathcal{M}_{n-1})]^{\mathrm{c}}) \leq \mu_\beta(\mathcal{N} \geq n) - \mu_\beta(\mathcal{W}^{\mathrm{c}}_\gamma) - \mu_\beta(\mathcal{X}^{\mathrm{c}}_R) - \mu_\beta(\mathcal{Z} > z).$$

Proposition 6.1 (ii) provides the sharp asymptotics of $\mu_\beta(\mathcal{N} \geq n)$. Choosing $\gamma > 0$, R large enough, and $z = n$, Lemmata 4.2 (ii), 4.1, 5.1 and Proposition 6.1 (i) imply that the remaining terms are negligible with respect to $\mu_\beta(\mathcal{N} \geq n)$. Hence

$$\mu_\beta([\mathcal{O}_{\rho_{k,\beta}}(\mathcal{M}_{n-1})]^{\mathrm{c}}) \leq \frac{1}{n!}\Big(\frac{\ell_\beta}{\bar{\ell}_\beta}\Big)^n (1 + o_\beta(1)). \tag{6.4}$$

For $k = 1$ this proves the upper bound in the statement. To deduce the lower bound we write

$$\begin{aligned}
&\mu_\beta(\mathcal{O}_{\rho_{2,\beta}}(\mathcal{M}_n) \setminus \mathcal{O}_{\rho_{1,\beta}}(\mathcal{M}_{n-1})) \\
&\quad = \mu_\beta([\mathcal{O}_{\rho_{1,\beta}}(\mathcal{M}_{n-1})]^{\mathrm{c}}) - \mu_\beta([\mathcal{O}_{\rho_{1,\beta}}(\mathcal{M}_{n-1})]^{\mathrm{c}} \cap (\mathcal{O}_{\rho_{2,\beta}}(\mathcal{M}_n))^{\mathrm{c}}) \\
&\quad \geq \mu_\beta([\mathcal{O}_{\rho_{1,\beta}}(\mathcal{M}_{n-1})]^{\mathrm{c}}) - \mu_\beta((\mathcal{O}_{\rho_{2,\beta}}(\mathcal{M}_n))^{\mathrm{c}}).
\end{aligned} \tag{6.5}$$

In view of (6.1), we can choose a sequence $(\delta_\beta)_{\beta>0}$ such that $\lim_\beta \delta_\beta \rho^{-p}_{1,\beta} = \infty$ and $\lim_\beta \delta_\beta \ell_\beta / T_\beta = 0$. For each $\gamma, R, z > 0$, by Lemma 6.3, eventually as $\beta \to \infty$, we have

$$\mu_\beta([\mathcal{O}_{\rho_{1,\beta}}(\mathcal{M}_{n-1})]^{\mathrm{c}}) \geq \mu_\beta(\mathcal{N} = n, \mathcal{B}_{n,\delta_\beta}) - \mu_\beta(\mathcal{W}^{\mathrm{c}}_\gamma) - \mu_\beta(\mathcal{X}^{\mathrm{c}}_R) - \mu_\beta(\mathcal{Z} > z).$$

Proposition 6.1 (iii) provides the sharp asymptotics of $\mu_\beta(\mathcal{N} = n, \mathcal{B}_{n,\delta_\beta})$. Choosing $\gamma > 0$, R large enough, and $z = n$, Lemmata 4.2 (ii), 4.1, 5.1 and Proposition 6.1 (i) imply that the remaining terms are negligible with respect to $\mu_\beta(\mathcal{N} = n, \mathcal{B}_{n,\delta_\beta})$. Hence,

$$\mu_\beta([\mathcal{O}_{\rho_{1,\beta}}(\mathcal{M}_{n-1})]^{\mathrm{c}}) \geq \frac{1}{n!}\Big(\frac{\ell_\beta}{\bar{\ell}_\beta}\Big)^n (1 + o_\beta(1)).$$

Recalling (6.5) and using (6.4) with $k = 2$ and n in place of $n-1$, the lower bound of the statement follows. □

Proof of Proposition 6.1. Proof of (i). Since $\{\mathcal{Z} \geq n\} \in \sigma(\{X_t\}_{t\in[T_\beta,\ell_\beta-T_\beta]})$, by Lemma 5.2 it is enough to prove the statement for $\mathbb{P}^{\beta}_{\pi_\beta}$ instead of μ_{β,ℓ_β}. This will be achieved by induction. More precisely, set $T_{0,\beta} = T_\beta$ and fix sequences $T_{1,\beta}, \ldots, T_{n,\beta}$ such that $T_{k,\beta}/T_{k-1,\beta} \to \infty$, $k = 1, \ldots, n$, and $T_{n,\beta}/\ell_\beta \to 0$. We recursively define the sequence of stopping times $\sigma_0, \ldots, \sigma_n$ on $C([0,\infty))$, by setting $\sigma_0 = 0$ and $\sigma_k = \inf\{t \geq T_\beta + \sigma_{k-1} : X_t = 0\}$, $k = 1, \ldots, n$. We claim that, for $k = 1, \ldots, n$,

$$\mathbb{P}^{\beta}_{\pi_\beta}(\sigma_k < t) \sim \frac{(tA_{\mathrm{W}}\sqrt{\beta})^k}{k!}\mathrm{e}^{-k\beta C_{\mathrm{W}}} \quad \text{uniformly for } t \in [T_{k,\beta}, \ell_\beta]. \tag{6.6}$$

Since $T_{1,\beta}/T_\beta \to \infty$, the case $k = 1$ follows from Theorem 4.5. To prove the inductive step, by the strong Markov property,

$$\mathbb{P}^{\beta}_{\pi_\beta}(\sigma_{k+1} < t) = \int_{T_\beta}^{t-T_\beta} \mathbb{P}^{\beta}_{\pi_\beta}(\sigma_k \in \mathrm{d}s)\mathbb{P}^{\beta}_{0}(\sigma_1 < t - s).$$

By Lemma 5.2 and the recursive hypothesis, it is enough to show

$$\int_{T_\beta}^{t-T_\beta} \mathbb{P}^{\beta}_{\pi_\beta}(\sigma_k \in \mathrm{d}s)\mathbb{P}^{\beta}_{\pi_\beta}(\tau < t - s - T_\beta) \sim \frac{(tA_{\mathrm{W}}\sqrt{\beta})^{k+1}}{(k+1)!}\mathrm{e}^{-(k+1)\beta C_{\mathrm{W}}},$$

uniformly for $t \in [T_{k+1,\beta}, \ell_\beta]$. We first observe that

$$\begin{aligned}\textstyle\int_{t-2T_\beta}^{t-T_\beta} \mathbb{P}^{\beta}_{\pi_\beta}(\sigma_k \in \mathrm{d}s)\mathbb{P}^{\beta}_{\pi_\beta}(\tau < t - s - T_\beta) &\leq \mathbb{P}^{\beta}_{\pi_\beta}(\sigma_k < t - T_\beta)\mathbb{P}^{\beta}_{\pi_\beta}(\tau \leq T_\beta)\\ &\sim \tfrac{T_\beta t^k (A_{\mathrm{W}}\sqrt{\beta})^{k+1}}{k!}\mathrm{e}^{-(k+1)\beta C_{\mathrm{W}}},\end{aligned}$$

where we used Theorem 4.5 and the recursive hypothesis. On the other hand, by the same theorem,

$$\begin{aligned}&\int_{T_\beta}^{t-2T_\beta} \mathbb{P}^{\beta}_{\pi_\beta}(\sigma_k \in \mathrm{d}s)\mathbb{P}^{\beta}_{\pi_\beta}(\tau < t - s - T_\beta)\\ &\quad= A_{\mathrm{W}}\sqrt{\beta}\mathrm{e}^{-\beta C_{\mathrm{W}}} \int_{T_\beta}^{t-2T_\beta} \mathbb{P}^{\beta}_{\pi_\beta}(\sigma_k \in \mathrm{d}s)(t - s - T_\beta)(1 + R_\beta(t - s)),\end{aligned}$$

with

$$\lim_{\beta\to\infty} \sup_{t\in[3T_\beta,\ell_\beta]} \sup_{s'\in[2T_\beta,t-T_\beta]} |R_\beta(s')| = 0.$$

By integration by parts, since $\mathbb{P}^{\beta}_{\pi_\beta}(\sigma_k \leq T_\beta) = 0$,

$$\begin{aligned}&\int_{T_\beta}^{t-2T_\beta} \mathbb{P}^{\beta}_{\pi_\beta}(\sigma_k \in \mathrm{d}s)(t - s - T_\beta)\\ &\quad= \int_{T_\beta}^{T_{k,\beta}} \mathrm{d}s\, \mathbb{P}^{\beta}_{\pi_\beta}(\sigma_k \leq s) + \int_{T_{k,\beta}}^{t-2T_\beta} \mathrm{d}s\, \mathbb{P}^{\beta}_{\pi_\beta}(\sigma_k \leq s) + \mathbb{P}^{\beta}_{\pi_\beta}(\sigma_k \leq t - 2T_\beta)T_\beta.\end{aligned}$$

By the recursive hypothesis and the choice of $T_{k,\beta}$,

$$\int_{T_{k,\beta}}^{t-2T_\beta} \mathrm{d}s\, \mathbb{P}^\beta_{\pi_\beta}(\sigma_k \le s) \sim \frac{(A_\mathrm{W}\sqrt{\beta})^k \mathrm{e}^{-k\beta C_\mathrm{W}}}{(k+1)!}\Big[(t-2T_\beta)^{k+1} - T_{k,\beta}^{k+1}\Big],$$

while

$$\int_{T_\beta}^{T_{k,\beta}} \mathrm{d}s\, \mathbb{P}^\beta_{\pi_\beta}(\sigma_k \le s) \le T_{k,\beta}\mathbb{P}^\beta_{\pi_\beta}(\sigma_k \le T_{k,\beta}) \sim \frac{(A_\mathrm{W}\sqrt{\beta})^k \mathrm{e}^{-k\beta C_\mathrm{W}}}{k!} T_{k,\beta}^{k+1},$$

and

$$\mathbb{P}^\beta_{\pi_\beta}(\sigma_k \le t-2T_\beta)T_\beta \sim T_\beta(t-2T_\beta)^k \frac{(A_\mathrm{W}\sqrt{\beta})^k \mathrm{e}^{-k\beta C_\mathrm{W}}}{k!}.$$

Gathering the previous bounds, (6.6) follows.
Proof of (ii). By item (i) it is enough to show that

$$\mu_{\beta,\ell_\beta}(\mathcal{N}=n, \mathcal{Z}=n) \sim \frac{1}{n!}\Big(\frac{\ell_\beta}{\bar{\ell}_\beta}\Big)^n.$$

Given $t_1 < t_2$ and $x, y \in \mathbb{R}\cup\{\mathrm{f}\}$ denote by $\mu^{x,y}_{\beta,[t_1,t_2]}$ the ϕ^4_1 measure on the interval $[t_1,t_2]$ with Dirichlet boundary conditions x, y at the endpoints. When x or y equals f we understand free boundary conditions. In particular $\mu^{\mathrm{f},\mathrm{f}}_{\beta,[0,\ell]} = \mu_{\beta,\ell}$. By the strong Markov property of $\mu_{\beta,\ell}$

$$\mu_{\beta,\ell}(\cdot|\mathcal{Z}=n) = \int \nu^n_{\beta,\ell}(\mathrm{d}t_1,\dots,\mathrm{d}t_n)\, \mu^{\mathrm{f},0}_{\beta,[0,t_1]} \otimes \mu^{0,0}_{\beta,[t_1,t_2]} \otimes \cdots \otimes \mu^{0,0}_{\beta,[t_{n-1},t_n]} \otimes \mu^{0,\mathrm{f}}_{\beta,[t_n,\ell]},$$

where

$$\nu^n_{\beta,\ell}(\mathrm{d}t_1,\dots,\mathrm{d}t_n) = \mu_{\beta,\ell}(\tau_1 \in \mathrm{d}t_1,\dots,\tau_n\in\mathrm{d}t_n|\mathcal{Z}=n).$$

Since the probabilities $\mu^{0,0}_{\beta,[t_k,t_{k+1}]}, k=1,\dots,n-1, \mu^{\mathrm{f},0}_{\beta,[0,t_1]}$, and $\mu^{0,\mathrm{f}}_{\beta,[t_n,\ell]}$ are invariant with respect to $X\mapsto -X$ we deduce $\mu_{\beta,\ell_\beta}(\mathcal{N}=n,\mathcal{Z}=n) = 2^{-n}\mu_{\beta,\ell_\beta}(\mathcal{Z}=n)$. The statement now follows from (i).
Proof of (iii). By the symmetry argument used in (ii), it is enough to show

$$\mu_{\beta,\ell_\beta}(\{\mathcal{Z}=n\}\cap\mathcal{B}_{n,\delta_\beta}) \sim \frac{2^n}{n!}\Big(\frac{\ell_\beta}{\bar{\ell}_\beta}\Big)^n.$$

This is achieved by the induction argument used in (i), noticing that Theorem 4.5 implies

$$\mathbb{P}^\beta_{\pi_\beta}(\ell_\beta\delta < \sigma_1 < t) \sim \frac{2\,t}{\bar{\ell}_\beta},$$

with σ_1 as introduced in (i). □

7. Convergence of the Transition Times to a Poisson Point Process

Let $(T_\beta)_{\beta>0}$ be a sequence meeting the first two conditions in (6.1) and recursively define the sequence of stopping times $\sigma_0, \sigma_1, \dots$ on $C([0,\infty))$, by setting $\sigma_0 = 0$ and $\sigma_k = \inf\{t \geq T_\beta + \sigma_{k-1} \colon X_t = 0\}$, $k \in \mathbb{N}$. Further set

$$\mathcal{N} = \inf\{k \in \mathbb{N} \colon X_{T_\beta+\sigma_{k-1}} X_{T_\beta+\sigma_k} < 0\},$$

and $\zeta = \sigma_{\mathcal{N}}$.

Lemma 7.1. *When X is sampled according to $\mathbb{P}^\beta_{\pi_\beta}$ and $\beta \to \infty$ the sequence of random variables $(\bar{\ell}_\beta^{-1}\zeta)_{\beta>0}$ converges in law to an exponential random variable with parameter one.*

Proof. By the strong Markov property and the symmetry of $\mathbb{P}^\beta_0$ with respect to $X \mapsto -X$, the random variable $\mathcal{N}$ has a geometric distribution with parameter $1/2$. Again by the strong Markov property the random variables $(\sigma_k - \sigma_{k-1})_{k\in\mathbb{N}}$ are independent and independent from $\mathcal{N}$. By Theorem 4.6 and the second statement of Lemma 5.2, for each $k \in \mathbb{N}$, as $\beta \to \infty$, the sequence of random variables $(\bar{\ell}_\beta^{-1}(\sigma_k - \sigma_{k-1}))_{\beta>0}$ converges in law to an exponential random variable with parameter two. The statement is achieved recalling that a geometric sum of independent exponential random variables is an exponential random variable. □

Proof of Theorem 2.3. We need to show that for each $R > 0$ the restriction of $\mu_{\beta,\ell_\beta} \circ \iota^{-1}_{\bar{\ell}_\beta}$ to $L^p((-R,R))$ converges weakly to the restriction of $\bar{\mu}$ to $L^p((-R,R))$. Recalling (2.14), by the first statement of Lemma 5.2, it is enough to show the statement with $\mathbb{P}^\beta_{\pi_\beta}$ instead of μ_{β,ℓ_β}.

Step 1. Recalling the sequence of stopping times (σ_k) introduced at the beginning of this section, for each $R, \delta > 0$,

$$\lim_{\beta\to\infty} \mathbb{P}^\beta_{\pi_\beta}\Big(\frac{1}{\bar{\ell}_\beta}\int_0^{R\bar{\ell}_\beta} \mathrm{d}t \,\Big|X_t - \sum_{k=0}^\infty \mathrm{sign}(X_{T_\beta+\sigma_k})\mathbf{1}_{[\sigma_k,\sigma_{k+1})}(t)\Big|^p > \delta\Big) = 0,$$

where we understand $\mathrm{sign}(0) = 0$.

Arguing as in the proof of Lemma 6.2, this step follows from Lemmata 4.1, 4.2, Theorem 4.6 and the strong Markov property.

Step 2. Sampling X according to $\mathbb{P}^\beta_{\pi_\beta}$, we recursively define the random variables $\mathcal{N}_j$, $j = 0, 1, \dots$, by setting $\mathcal{N}_0 = 0$ and

$$\mathcal{N}_j = \inf\{k > \mathcal{N}_j \colon X_{T_\beta+\sigma_{k-1}} X_{T_\beta+\sigma_k} < 0\} - \mathcal{N}_{j-1}.$$

Let also $\zeta_j = \sigma_{\mathcal{N}_j}$. Then the random variables $(\zeta_j - \zeta_{j-1})_j$ are independent. Moreover, for each j, $\bar{\ell}_\beta^{-1}(\zeta_j - \zeta_{j-1})$ converges in law to an exponential random variable of parameter one in the limit $\beta \to \infty$. This step follows directly from the strong Markov property and Lemma 7.1.

The statement is achieved by combining Steps 1 and 2 together with the symmetry of $\mathbb{P}^\beta_{\pi_\beta}$ with respect to $X \mapsto -X$. □

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References

1. Alberti, G.: Variational models for phase transitions, an approach via Γ-convergence. In: Buttazzo, G., Marino, A., Murthy, M.K.V. (eds.) Calculus of Variations and Partial Differential Equations: Topics on Geometrical Evolution Problems and Degree Theory, pp. 95–114. Springer, Berlin (2000)
2. Berglund, N., Gentz, B.: Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers' law and beyond. *Electron. J. Probab.* **18**(24), 1–58, 2013
3. Berglund, N., Di Gesù, G., Weber, H.: An Eyring-Kramers law for the stochastic Allen-Cahn equation in dimension two. *Electron. J. Probab.* **22**, 1–27, 2017
4. Bertini, L., Brassesco, S., Buttà, P.: Soft and hard wall in a stochastic reaction diffusion equation. *Arch. Ration. Mech. Anal.* **190**, 307–345, 2008
5. Bertini, L., Brassesco, S., Buttà, P.: Dobrushin states in the ϕ^4_1 model. *Arch. Ration. Mech. Anal.* **190**, 477–516, 2008
6. Billingsley, P.: Convergence of Probability Measures. Wiles and Sons, New York (1968)
7. Brassesco, S., Buttà, P.: Interface fluctuations for the $d = 1$ stochastic Ginzburg–Landau equation with non-symmetric reaction term. *J. Stat. Phys.* **93**, 1111–1142, 1998
8. Brassesco, S., Masi, A., Presutti, E.: Brownian fluctuations of the interface in the $d = 1$ Ginzburg–Landau equation with noise. *Ann. Inst. H. Poincaré* **31**, 81–118, 1995
9. Brooks, M., Di Gesù, G.: Sharp tunneling estimates for a double-well model in infinite dimension. *J. Funct. Anal.* **281**, 109029, 2021
10. Cassandro, M., Orlandi, E., Presutti, E.: Interfaces and typical Gibbs configurations for one-dimensional Kac potentials. *Probab. Theory Relat. Fields* **96**, 57–96, 1993
11. Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: Schrödinger Operators with Application to Quantum Mechanics and Global Geometry, Text and Monographs in Physics. Springer, Berlin (1987)

12. Davies, E.B., Simon, B.: Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.* **59**, 335–395, 1984
13. Dembo, A., Zeitouni, O.: Large Deviations Techniques and Applications, 2nd edn. Springer, New York (1998)
14. Di Gesù, G., Lelièvre, T., Le Peutrec, D., Nectoux, B.: Sharp asymptotics of the first exit point density. *Ann. PDE* **5**, 5, 2019
15. Di Gesù, G., Mariani, M.: Full metastable asymptotic of the Fisher Information. *SIAM J. Math. Anal.* **49**, 3048–3072, 2017
16. Di Gesù, G.: Spectral analysis of discrete metastable diffusions. *Commun. Math. Phys.* **402**, 543–580, 2023
17. Freidlin, M.I., Wentzell, A.D.: Random Perturbations of Dynamical Systems, 2nd edn. Springer, Berlin (1998).**Translated from the 1979 Russian original by Joseph Szücs**.
18. Funaki, T.: The scaling limit for a stochastic PDE and the separation of phases. *Prob. Theory Relat. Fields* **102**, 221–288, 1995
19. Harrell, E.M.: On the rate of asymptotic eigenvalue degeneracy. *Commun. Math. Phys.* **60**, 73–95, 1978
20. Harrell, E.M.: Double Wells. *Commun. Math. Phys.* **75**, 239–261, 1980
21. Helffer, B.: Introduction to the Semiclassical Analysis for the Schrödinger Operator and Applications. Lecture Notes in Math, vol. 1336. Springer (1988)
22. Helffer, B., Sjöstrand, J.: Multiple wells in the semiclassical limit I. *Commun. Partial Differ. Equ.* **9**, 337–408, 1984
23. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge (2012)
24. Jona-Lasinio, G., Martinelli, F., Scoppola, E.: New approach to the semiclassical limit of quantum mechanics. *Commun. Math. Phys.* **80**, 223–254, 1981
25. Kim, S., Seo, I.: Metastability of Ising and Potts models without external fields in large volumes at low temperatures. *Commun. Math. Phys.* **396**, 383–449, 2022
26. Kipnis, C., Landim, C.: Scaling Limits of Interacting Particle Systems. Springer, New York (1999)
27. Landim, C.: Metastable Markov chains. *Probab. Surv.* **16**, 143–227, 2019
28. Modica, L., Mortola, S.: Un esempio di Γ-convergenza. *Boll. Un. Mat. Ital. B* **14**, 285–29, 1977
29. Mariani, M.: Large deviations principles for stochastic scalar conservation laws. *Probab. Theory Relat. Fields* **147**, 607–648, 2010
30. Otto, F., Weber, H., Westdickenberg, M.: Invariant measure of the stochastic Allen–Cahn equation: the regime of small noise and large system size. *Electron. J. Probab.* **19**, 1–76, 2014
31. Peletier, M., Savaré, G., Veneroni, M.: From diffusion to reaction via Gamma-convergence. *SIAM J. Math. Anal.* **42**, 1805–1825, 2010
32. Vanden-Eijnden, E., Westdickenberg, M.G.: Rare events in stochastic partial differential equations on large spatial domains. *J. Stat. Phys.* **131**, 1023–1038, 2008
33. Rezakhanlou, F., Seo, I.: Scaling limit of small random perturbation of dynamical systems. *Ann. Inst. H. Poincaré Probab. Stat.* **59**, 867–903, 2023
34. Schonmann, R.H., Tanaka, N.I.: One-dimensional caricature of phase transition. *J. Stat. Phys.* **61**, 241–252, 1990
35. Simon, B.: Semi-classical analysis of low lying eigenvalues I. Non-degenerate minima: asymptotic expansion. *Ann. l'I.H.P. Phys. théor.* **38**, 295–308, 1983
36. Simon, B.: Semiclassical analysis of low lying eigenvalues. II. *Tunnel. Ann. Math.* **120**, 89–118, 1984
37. Simon, B.: Semiclassical analysis of low lying eigenvalues. IV. The flea on the elephant. *J. Funct. Anal.* **63**, 123–136, 1985
38. Simon, B.: Functional Integration and Quantum Physics, 2nd edn. AMS Chelsea Publishing, Providence (2005)

39. Van der Waals, D.J.: The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density. *J. Stat. Phys*. **20**, 200–244, 1979 (English translation of the 1863 original)
40. Weber, H.: Sharp interface limit for invariant measures of a stochastic Allen–Cahn equation. *Commun. Pure Appl. Math.* **63**, 1071–1109, 2010

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