SIAM J. MATH. ANAL.
Vol. 57, No. 5, pp. 5746–5770

LARGE TIME ANALYSIS OF THE RATE FUNCTION ASSOCIATED TO THE BOLTZMANN EQUATION: DYNAMICAL PHASE TRANSITIONS*

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Abstract. We analyze the large time behavior of the rate function that describes the probability of large fluctuations of an underlying microscopic model associated with the homogeneous Boltzmann equation, such as the Kac walk. We consider in particular the asymptotic of the number of collisions per particle and per unit of time and show it exhibits a phase transition in the joint limit in which the number of particles N and the time interval [0,T] diverge. More precisely, due to the existence of Lu–Wennberg solutions, the corresponding limiting rate function vanishes for subtypical values of the number of collisions. We also analyze the second order large deviations showing that the probability of subtypical fluctuations is exponentially small in N, independently of T. As a key point, we establish the controllability of the homogeneous Boltzmann equation.

Key words. Boltzmann equation, large deviations, dynamical phase transitions, Kac model

MSC codes. 35Q20, 60F10, 82C40

DOI. 10.1137/24M1719165

1. Introduction. The limiting behavior of many body systems in the low density regime is described by kinetic equations. We focus on the spatially homogeneous case, for which the typical behavior is described by the homogeneous Boltzmann equation for the one-particle velocity distribution. Recently, some progress has been achieved in the analysis of atypical behavior in the context of both deterministic and stochastic microscopic dynamics. In particular, after [17, 22], that large deviations result over a finite time window in the limit of infinitely many particles has been proven in [1, 2, 3, 7, 14].

As a basic microscopic model, we consider a version of Kac's walk with hard sphere collision kernel, which is a jump Markov process on the state space given by N velocities in $\mathbb{R}^d$. The dynamics are specified in terms of binary collisions which conserve particle number, momentum, and kinetic energy; the collision rate is chosen in such a way that the typical number of collisions per particle in the unit of time is of order one, formalized by the generator (2.1) below. As a classic result, as the number of particle N diverges, the empirical measure converges to the solution to the homogeneous Boltzmann equation [23].

We focus on the behavior of the number of collisions in the joint limit in which the number of particles N and the time window $[0,T]$ diverge.

*Received by the editors December 18, 2024; accepted for publication (in revised form) July 22, 2025; published electronically October 7, 2025.
https://doi.org/10.1137/24M1719165
Funding: The work of the first and second authors was supported by PRIN grant 202277WX43 Emergence of condensation-like phenomena in interacting particle systems: kinetic and lattice models, founded by the European Union - Next Generation EU. The work of the first and third authors was supported by the European Union (ERC CoG KiLiM) project 101125162. The work of the fourth author was supported by the Royal Commission for the Exhibition of 1851.
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Letting $q_{N,T}$ be the number of collisions per particle and per unit of time, as argued in (2.14), in either double limit $N \to \infty, T \to \infty$, $q_{N,T} \to \bar{q}_e := \frac{1}{2}\int B(v - v_*, \omega) M_e(\mathrm{d}v) M_e(\mathrm{d}v_*)\, \mathrm{d}\omega$, where M_e is the Maxwellian of energy e prescribed by the initial conditions, $\mathrm{d}\omega$ is the Haar measure on the sphere, and B is the collision kernel. Note that if B does not depend on the velocities, as in the case of Maxwellian molecules, then $\bar{q}_e$ does not depend on e. In fact, the statistic of $q_{N,T}$ is described by a Poisson random variable.

For a fixed time intervall $[0, T]$, the probability of fluctuations of $q_{N,T}$ as N diverges can be obtained by projecting the rate function associated to the microscopic dynamics deduced in [2, 14] in the microcanonical setting. Here, we analyze the limit of such projection as T diverges, thus completing the analysis of the large deviation in the number of collisions per particle and per unit of time. We remark that the present analysis does not rely on the details of the microscopic dynamics but only on the rate function; in this respect, it can also be applied to the case of Newtonian dynamics [7].

For the asymptotic behavior of $q_{N,T}$, a special role is played by the so-called Lu–Wennberg solutions to the homogeneous Boltzmann equation [18], which are weak solutions with increasing energy. A class of these solutions can be obtained from the underlying microscopic dynamics by considering only fluctuations of the initial distribution; therefore, their cost does not depend on the time interval. By exploiting such Lu–Wennberg solutions, we deduce that the probability of subtypical fluctuations in $q_{N,T}$ is exponentially small in N but independent on T. In contrast, to construct supertypical fluctuations, it is necessary to change the dynamics; therefore, the corresponding probability is exponentially small in NT.

In summary, denoting by $\mathbb{P}_e^N$ the distribution induced by the microscopical dynamic with energy per particle given by e, we show that

$$\mathbb{P}_e^N (q_{N,T} \approx q) \sim \mathrm{e}^{-NT i_e(q)},$$

where i_e is a convex function on $[0, \infty)$, vanishing on $[0, \bar{q}_e]$, and strictly positive on $(\hat{q}_e, \infty)$ for some slightly larger $\hat{q}_e > \bar{q}_e$. This dichotomy has the canonical interpretation [4, 9, 11] in macroscopic fluctuation theory as the existence of a *dynamical phase transition* for the collision numbers $q_{N,T}$.

We do not obtain a explicit expression for i_e, but we provide explicit upper and lower bounds which are sufficient to show the behavior described in words above. The upper bound is obtained by choosing a time-independent path, where the single time velocity distribution is characterized by a static variational problem whose solution is not Maxwellian. The lower bound is instead obtained by comparison with a suitable static strategy. We expect that neither the upper nor the lower bound is optimal, and in fact the optimal path should exhibit nontrivial time dependence.

The second order asymptotic can be formalized as

$$\mathbb{P}_e^N (q_{N,T} \approx q) \sim \mathrm{e}^{-N j_e(q)}.$$

Clearly, $j_e(q) = +\infty$ for $q > \bar{q}_e$. In the case in which the initial microscopic distribution is the uniform measure on the energy surface, we obtain an explicit expression for j_e related to the relative entropy between two Maxwellians.

The present analysis requires two auxiliary results of independent interest which we next briefly describe. Given a path with a finite rate function, we prove the chain rule for its entropy. While for the energy conserving path this statement has been proven in [12], we here show that it holds also for a path for which the energy is not constant, as in the case of Lu–Wennberg solutions. The second result regards the controllability of the homogeneous Boltzmann equation. More precisely, given two distributions with bounded energy and entropy, we show that they can be connected by a path with finite cost.

In the context of hydrodynamic scaling limits, an analogous problem to the one we here consider is the fluctuation in the total current, which exhibits interesting behavior [5, 4, 6]. The presence of two different scaling regimes for subtypical and supertypical fluctuations in the total number of jumps has been proven for the so-called east model [8, 9], which has been also interpreted as a dynamical phase transition. We finally refer to [13] for the asymptotic of the total number of jumps in the context of nonlinear Markov processes.

2. Background and results.

We first recall the analysis in [2, 14] which describes the large deviations asymptotics for the empirical measure and flow of the Kac walk over a fixed time interval $[0,T]$ in the limit of infinitely many particles, $N\to\infty$. This result yields, by projection, the large deviations principle of the total number of collisions per particle in this limit. The corresponding rate functional is expressed by a time-dependent variational problem. By analyzing this variational problem in the limit $T\to\infty$, we then show that the total number of collisions per particle and per unit of time exhibits a dynamical phase transition in the joint limit $N,T\to\infty$.

Microscopic model and empirical observables. Fix $d\geq 2$ and set $\Sigma^N = \left(\mathbb{R}^d\right)^N$. We consider the Kac walk given by the Markov process on the configuration space Σ^N whose generator acts on bounded continuous functions $f\colon \Sigma^N\to\mathbb{R}$ as

$$\mathcal{L}_N f(\boldsymbol{v}) = \frac{1}{N}\sum_{\{i,j\}} L_{i,j} f(\boldsymbol{v}), \tag{2.1}$$

where the sum is carried over the unordered pairs $\{i,j\}\subset\{1,..,N\}$, $i\neq j$, and

$$L_{i,j} f(\boldsymbol{v}) = \int_{\mathbb{S}_{d-1}} \mathrm{d}\omega\, B(v_i - v_j,\omega)\big[f\big(T^{\omega}_{i,j}\boldsymbol{v}\big) - f(\boldsymbol{v})\big].$$

Here, $\mathbb{S}_{d-1}$ is the sphere in $\mathbb{R}^d$, and the postcollisional vector of velocities is given by

$$\big(T^{\omega}_{i,j}\boldsymbol{v}\big)_k = \begin{cases} v_i + (\omega\cdot(v_j - v_i))\omega & \text{if } k=i\\ v_j - (\omega\cdot(v_j - v_i))\omega & \text{if } k=j\\ v_k & \text{otherwise,}\end{cases} \tag{2.2}$$

and the collision kernel B is given by

$$B(v - v_*,\omega) = \frac{1}{2}|(v - v_*)\cdot\omega|. \tag{2.3}$$

The collisional dynamics preserve the total particle number, momentum, and energy, given by the integrals of

$$\boldsymbol{\zeta}\colon \mathbb{R}^d\mapsto[0,+\infty)\times\mathbb{R}^d, \qquad \boldsymbol{\zeta} = (\zeta_0,\zeta)(v) = (|v|^2/2, v).$$

In the sequel, we fix $e\in(0,\infty)$ and consider the restriction of the Kac walk to the set

$$\Sigma^N_e := \left\{\boldsymbol{v}\in\Sigma^N\colon\ \frac{1}{N}\sum_{i=1}^N \boldsymbol{\zeta}(v_i) = (e,0)\right\}. \tag{2.4}$$

By Galilean invariance, the choice of vanishing total momentum is not special, and any other choice can be achieved by selecting a suitable frame of reference. In constrast, the parameter e, which represents the energy per particle, will play an important role in our analysis. We denote by $(\boldsymbol{v}(t))_{t\geq 0}$ the Markov process generated by $\mathcal{L}_N$ which is, by direct computation, ergodic and reversible with respect to the uniform probability on Σ_e^N. Given $T>0$ and a probability ν on Σ_e^N, the dynamics carry the underlying probability measure to the law of the Kac process $\mathbb{P}_\nu^N$ on the Skorokhod space $D([0,T];\Sigma_e^N)$.

Let $\mathscr{P}(\mathbb{R}^d)$ be the set of probability measures π on $\mathbb{R}^d$ equipped with weak topology. We denote by $\mathscr{P}_e$ the subset of $\mathscr{P}(\mathbb{R}^d)$ given by the probabilities with vanishing mean and second moment bounded by $2e$, namely, the set of $\pi\in\mathscr{P}(\mathbb{R}^d)$ such that $\pi(\zeta_0)\leq e$ and $\pi(\zeta)=0$. $\mathscr{P}_e$ is a compact convex subset of $\mathscr{P}(\mathbb{R}^d)$, and we equip it with the relative topology and the corresponding Borel σ-algebra. For $T>0$, let $D\big([0,T];\mathscr{P}_e\big)$ be the space of $\mathscr{P}_e$-valued càdlàg paths endowed with the Skorokhod topology and the corresponding Borel σ-algebra. The *empirical measure* is the map $\pi^N\colon \Sigma_e^N\to\mathscr{P}_e$ defined by

$$\pi^N(\boldsymbol{v}) := \frac{1}{N}\sum_{i=1}^N \delta_{v_i}. \tag{2.5}$$

We denote by $\boldsymbol{\pi}^N$ the map from $D\big([0,T];\Sigma_e^N\big)$ to $D\big([0,T];\mathscr{P}_e\big)$ defined by $\boldsymbol{\pi}_t^N(\boldsymbol{v}):=\pi^N(\boldsymbol{v}(t))$, $t\in[0,T]$. Given suitable assumptions on the initial conditions, as N tends to infinity, by propagation of chaos [23, 20], the family $\boldsymbol{\pi}^N(\boldsymbol{v})$ converges to the unique energy-conserving solution to the spatially homogeneous Boltzmann equation

$$\partial_t f_t(v) = \int_{\mathbb{R}^d} \mathrm{d}v_* \int_{S_{d-1}} \mathrm{d}\omega\, B(v-v_*,\omega)\big(f_t(v')f_t(v_*') - f_t(v)f_t(v_*)\big). \tag{2.6}$$

For a fixed time horizon $T>0$, we denote by $\mathscr{M}_T$ the subset of the finite measures $\boldsymbol{Q}$ on $[0,T]\times\mathbb{R}^{2d}\times\mathbb{R}^{2d}$ that satisfy $\boldsymbol{Q}(\mathrm{d}t;\mathrm{d}v,\mathrm{d}v_*,\mathrm{d}v',\mathrm{d}v_*')=\boldsymbol{Q}(\mathrm{d}t;\mathrm{d}v_*,\mathrm{d}v,\mathrm{d}v',\mathrm{d}v_*')=\boldsymbol{Q}(\mathrm{d}t;\mathrm{d}v,\mathrm{d}v_*,\mathrm{d}v_*',\mathrm{d}v')$, which we endow with the weak topology[1] and the corresponding Borel $\sigma-$algebra. The *empirical flow* is the map $\boldsymbol{Q}_{[0,T]}^N\colon D\big([0,T];\Sigma_e^N\big)\to\mathscr{M}_T$ defined by specifying for each bounded and continuous function $F\colon [0,T]\times\mathbb{R}^{2d}\times\mathbb{R}^{2d}\to\mathbb{R}$ satisfying

$$F(t;v,v_*,v',v_*') = F(t;v_*,v,v',v_*') = F(t;v,v_*,v_*',v')$$

the integral

$$\boldsymbol{Q}_{[0,T]}^N(\boldsymbol{v})(F) := \frac{1}{N}\sum_{\{i,j\}}\sum_{k\geq 1} F\big(\tau_k^{i,j}; v_i(\tau_k^{i,j}-), v_j(\tau_k^{i,j}-), v_i(\tau_k^{i,j}), v_j(\tau_k^{i,j})\big), \tag{2.7}$$

where $(\tau_k^{i,j})_{k\geq 1}$ are the jump times of the pair (v_i,v_j) in the time window $[0,T]$, and $v_i(t-)=\lim_{s\uparrow t} v_i(s)$ is the left-limit. In view of the conservation of energy and momentum, the measure $\boldsymbol{Q}_{[0,T]}^N(\mathrm{d}t;\cdot)$ is supported on the set of pre and postcollisional velocities satisfying

$$\big\{\boldsymbol{\zeta}(v)+\boldsymbol{\zeta}(v_*) = \boldsymbol{\zeta}(v')+\boldsymbol{\zeta}(v_*')\big\}\subset\mathbb{R}^{2d}\times\mathbb{R}^{2d}.$$

[1]I.e., the topology generated by the maps $Q\mapsto Q(F), F\in C_{\mathrm{b}}([0,T]\times\mathbb{R}^{2d}\times\mathbb{R}^{2d})$.

The rate function in a fixed time window. For $T > 0$, let $\mathscr{S}_{e,T}$ be the subset of $D\big([0,T];\mathscr{P}_e\big) \times \mathscr{M}_T$ given by elements $(\boldsymbol{\pi},\boldsymbol{Q})$ that satisfies, for each $\phi \in C_{\mathrm{b}}^1([0,T]\times\mathbb{R}^d)$, the balance equation

$$\pi_T(\phi_T) - \pi_0(\phi_0) - \int_0^T \mathrm{d}t\, \pi_t(\partial_t\phi_t) - \int \boldsymbol{Q}(\mathrm{d}t;\mathrm{d}v,\mathrm{d}v_*,\mathrm{d}v',\mathrm{d}v'_*)\bar\nabla\phi_t(v,v_*,v',v'_*), \tag{2.8}$$

where

$$\bar\nabla\phi(v,v_*,v',v'_*) = \phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*).$$

By conservation of the number of particles, for each $\boldsymbol{v} \in \Sigma_e^N$, $\mathbb{P}_{\boldsymbol{v}}^N$-almost surely, the pair $(\pi^N, \boldsymbol{Q}_{[0,T]}^N)$ belongs to the set $\mathscr{S}_{e,T}$.

The analysis in [2] provides the large deviation principle for the pair $\big(\pi^N, \boldsymbol{Q}_{[0,T]}^N\big) \in \mathscr{S}_{e,T}$ in the limit $N \to \infty$ with T fixed. In order to describe the corresponding rate function, which takes into account both the fluctuations due to the initial distribution of the velocities and ones due to the stochastic dynamics, we must first specify the initial distribution for the Kac walk. To this end, fix a probability $m \in \mathcal{P}(\mathbb{R}^d)$ satisfying the following conditions.

Assumption 2.1. There exists $\gamma_0^* \in (0,+\infty]$ such that
- (i) m is absolutely continuous with respect to the Lebesgue measure, and m is strictly positive on open sets,
- (ii) $m(\mathrm{e}^{\gamma_0\zeta_0}) < +\infty$ for any $\gamma_0 \in (-\infty,\gamma_0^*)$ and $\lim_{\gamma_0\uparrow\gamma_0^*} m(\mathrm{e}^{\gamma_0\zeta_0}) = +\infty$,
- (iii) for each $\boldsymbol{\gamma} = (\gamma_0,\gamma) \in (-\infty,\gamma_0^*)\times\mathbb{R}^d$, the Fourier transform of $\frac{\mathrm{d}m}{\mathrm{d}v}\mathrm{e}^{\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}}$ belongs to $L^1(\mathbb{R}^d)$, and
- (iv) there exists $c > 0$ such that $\frac{\mathrm{d}m}{\mathrm{d}v} \geq \frac{1}{c}\exp\{-c|v|^2\}$.

These conditions hold for the most important case when $m = M_e$, the Maxwellian with vanishing average velocity and average energy e, with $\gamma_0^* = d/(2e)$.

Following [2], we then consider the Kac walk with initial distribution of the velocities given by ν_e^N, the product measure $m^{\otimes N}$ conditioned on vanishing total momentum and energy per particle given by e. Such a conditional measure is well defined in view of the existence of a regular version of conditional probabilities, and ν_e^N is supported on Σ_e^N. Furthermore, by standard properties of Gaussian measures, if $m = M_e$, then ν_e^N is the uniform probability on Σ_e^N, which is a reversible invariant measure for the Kac process.

For notation convenience, we will hereafter also assume that m has vanishing average momentum and average energy e; namely, $m(\boldsymbol{\zeta}) = (e,0)$. This can be achieved by a suitable exponential tilt which does not affect the conditional probability ν_e^N.

For the choice ν_e^N of the initial distribution of the velocities, the static rate function $H_e(\cdot|m)\colon \mathscr{P}_e \to [0,+\infty]$ is the convex functional given by

$$H_e(\pi|m) := \mathrm{Ent}\;(\pi|m) + \gamma_0^*\big[e - \pi\big(\zeta_0\big)\big], \tag{2.9}$$

where $\mathrm{Ent}(\cdot|\cdot)$ is the relative entropy between two probability measures. This rate function describes the static large deviations of the empirical measure $\big\{\pi^N(\boldsymbol{v})\big\}_{N\geq 1}$ when $\boldsymbol{v} \in \Sigma_e^N$ is sampled according to ν_e^N. Note that $H_e(\pi|m)$ can be finite also when the energy of π is strictly smaller than e; that is, with probability exponentially small in N—but not super-exponentially small—some of the energy may "escape to infinity." In the case in which $m = M_e$ so that ν_e^N is the uniform probability on Σ_e^N, this functional has been originally derived in [16]. It reads as

$$H_e(\pi|M_e) = \int \mathrm{d}\pi \log \frac{\mathrm{d}\pi}{\mathrm{d}v} + \frac{d}{2}\left(\log \frac{4\pi e}{d} + 1\right). \tag{2.10}$$

In order to describe the dynamical contribution to the rate function, let $r(v,v^*;\cdot)$ be the measure on $\mathbb{R}^{2d}$ supported on $\{\boldsymbol{\zeta}(v) + \boldsymbol{\zeta}(v_*) = \boldsymbol{\zeta}(v') + \boldsymbol{\zeta}(v'_*)\}$ such that $r(v,v_*,\mathrm{d}v',\mathrm{d}v'_*) = \mathrm{d}\omega\, B(v-v_*,\omega)$, where v' and v'_* are related to ω by the collision rules, as in (2.2). For $\pi \in D\big([0,T];\mathscr{P}_e\big)$, let $\boldsymbol{Q}^\pi \in \mathscr{M}_T$ be the measure defined by

$$\boldsymbol{Q}^\pi(\mathrm{d}t;\mathrm{d}v,\mathrm{d}v_*,\mathrm{d}v',\mathrm{d}v'_*) := \frac{1}{2}\,\mathrm{d}t\,\pi_t(\mathrm{d}v)\pi_t(\mathrm{d}v_*)\,r(v,v_*;\mathrm{d}v',\mathrm{d}v'_*) \tag{2.11}$$

and observe that $\boldsymbol{Q}^\pi(\mathrm{d}t,\cdot)$ is supported on $\big\{\boldsymbol{\zeta}(v) + \boldsymbol{\zeta}(v_*) = \boldsymbol{\zeta}(v') + \boldsymbol{\zeta}(v'_*)\big\}$.

Let $\mathscr{S}^{\mathrm{ac}}_{e,T}$ be the subset of $\mathscr{S}_{e,T}$ given by the elements $(\boldsymbol{\pi},\boldsymbol{Q})$ such that $\pi \in C\big([0,T];\mathscr{P}_e\big)$ and $\boldsymbol{Q} \ll \boldsymbol{Q}^\pi$. The dynamical rate function $J_{e,T}\colon \mathscr{S}_{e,T} \to [0,\infty]$ is defined by

$$J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q}) := \begin{cases} \displaystyle\int \mathrm{d}\boldsymbol{Q}^\pi \left[\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}\boldsymbol{Q}^\pi}\log\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}\boldsymbol{Q}^\pi} - \left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}\boldsymbol{Q}^\pi} - 1\right)\right] & \text{if } (\boldsymbol{\pi},\boldsymbol{Q}) \in \mathscr{S}^{\mathrm{ac}}_{e,T}, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.12}$$

In [2], it is shown that when the initial distribution of the velocities for the Kac walk is given by ν_e^N, then the pair empirical measure and flow satisfies a large deviation upper bound with speed N and rate function $I_{e,T}(\cdot|m)\colon \mathscr{S}_{e,T} \to [0,\infty]$ given by

$$I_{e,T}((\boldsymbol{\pi},\boldsymbol{Q})|m) = H_e(\boldsymbol{\pi}_0|m) + J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q}). \tag{2.13}$$

where $H_e\colon \mathscr{P}_e \to [0,\infty]$ takes into account the fluctuations in the initial condition, while $J_{e,T}\colon \mathscr{S}_{e,T} \to [0,\infty]$ encodes the dynamical fluctuations. A matching lower bound is proven for neighborhoods of pairs $(\boldsymbol{\pi},\boldsymbol{Q})$ such that $\boldsymbol{Q}$ has a bounded second moment and for a class of Lu–Wennberg solutions [18].

Denote by $q_{N,T}$ the total number of collisions in the Kac walk per particle and per unit of time so that $NTq_{N,T}$ is the total number of collisions in the time window $[0,T]$. From the very definition of the empirical flow, $q_{N,T}$ can be obtained from the mass of $\boldsymbol{Q}^N_{[0,T]}$ and more precisely $q_{N,T} = T^{-1}\boldsymbol{Q}^N_{[0,T]}(1)$. We claim that $q_{N,T}$ satisfies the law of large numbers in probability with respect to $\mathbb{P}^N_{\nu_e^N}$ as follows:

$$\lim_{T\to+\infty}\lim_{N\to+\infty} q_{N,T} = \bar{q}_e = \lim_{N\to+\infty}\lim_{T\to+\infty} q_{N,T}, \tag{2.14}$$

where

$$\bar{q}_e := \frac{1}{2}\int M_e(\mathrm{d}v)M_e(\mathrm{d}v_*)B(v-v_*,\omega)\,\mathrm{d}\omega. \tag{2.15}$$

The first equality in (2.14) follows from the convergence of the particle dynamics to the homogeneous Boltzmann equation with initial datum m and the convergence of its solution to the Maxwellian. The second equality follows from the ergodicity of the microscopic dynamics and the convergence to the Maxwellian of the empirical measure when the velocities are sampled with respect to the uniform measure on Σ_e^N.

For fixed $T>0$, the contraction principle allows us to transfer the large deviation results from the empirical flow to the empirical collision number $\{q_{N,T}\}_{N\ge 1}$ in terms of the rate function $\mathscr{I}_{e,T}\colon [0,+\infty)\to[0,+\infty]$ given by

$$\mathscr{I}_{e,T}(q|m):=\inf\Big\{I_{e,T}\big((\boldsymbol{\pi},\boldsymbol{Q})\big|m\big),\ (\boldsymbol{\pi},\boldsymbol{Q})\in\mathscr{S}_{e,T} \text{ such that } \boldsymbol{Q}(1)=Tq\Big\}. \tag{2.16}$$

It follows from [2] that $\{q_{N,T}, N\ge 1\}$ satisfies a large deviation upper bound with speed N and this rate, but a matching lower bound would require an additional regularity for the optimal path for the variational problem on the right-hand side of (2.16). Note that, as already discussed in the notation, the rate function (2.16) depends on the choice of the probability m describing the initial distribution of the velocities.

Main results. The present purpose is to investigate the large time behavior of the rate function in (2.16) which has a nontrivial structure, exhibiting in particular two different scaling regimes at large and small, respectively, empirical collision numbers. As customary in large deviations theory [19], the relevant notion to describe the convergence of the functions in (2.16) is De Giorgi's Γ-convergence.

To describe the first scaling regime, we first introduce the limiting rate function. Fix $\mu\in\mathscr{P}_e$ with $H_e(\mu|M_e)<+\infty$, and let $i_e(q|\mu)\colon (0,+\infty)\to[0,+\infty]$ be the function defined by

$$i_e(q|\mu):=\inf_{T>0}\frac{1}{T}\inf_{(\boldsymbol{\pi},\boldsymbol{Q})\in\mathscr{A}_{e,T}(q|\mu)} J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q}), \tag{2.17}$$

where

$$\mathscr{A}_{e,T}(q|\mu)=\Big\{(\boldsymbol{\pi},\boldsymbol{Q})\in\mathscr{S}_{e,T}:\pi_T=\pi_0=\mu,\boldsymbol{Q}(1)=Tq\Big\}.$$

We extend $i_e(\cdot|\mu)$ to a function on $[0,+\infty)$ by setting

$$i_e(0|\mu):=\liminf_{q\downarrow 0} i_e(q|\mu). \tag{2.18}$$

Proposition 2.2. *The function $i_e(\cdot|\mu)$ does not depend on μ, is continuous, and is convex on $[0,+\infty)$. Furthermore,*

$$i_e(q|\mu)=\lim_{T\to\infty}\frac{1}{T}\inf_{(\boldsymbol{\pi},\boldsymbol{Q})\in\mathscr{A}_{e,T}(q|\mu)} J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q}) \tag{2.19}$$

and vanishes on the interval $[0,\bar{q}_e]$.

In view of this result, here and after we drop the dependence on μ in the notation for i_e.

Theorem 2.3. *Fix $m\in\mathscr{P}_e$ meeting the conditions in Assumption 2.1. The sequence of functions $\big\{T^{-1}\mathscr{I}_{e,T}(\cdot|m)\big\}_{T>0}$ on $[0,+\infty)$, defined in (2.16), is equi-coercive, and Γ-converges to i_e as $T\to\infty$. Namely,*

(i) *for each $\ell>0$, there is a compact $K_\ell\subset\subset[0,+\infty)$ such that $\big\{q\in[0,+\infty)\colon\ T^{-1}\mathscr{I}_{e,T}(\cdot|m)\le\ell\big\}\subset K_\ell$ eventually as $T\to\infty$,*

(ii) *for each $q\in[0,+\infty)$ and each sequence $q_T\to q$, the inequality $\liminf_{T\to\infty}T^{-1}\mathscr{I}_{e,T}(q_T|m)\ge i_e(q)$ holds, and*

(iii) *for each $q\in[0,+\infty)$, there exists a sequence $q_T\to q$ such that $\limsup_{T\to\infty}T^{-1}\mathscr{I}_{e,T}(q_T|m)\le i_e(q)$.*

In view of standard property of Γ-convergence, see, e.g. [19], this statement together with the large deviations upper bound with fixed T in [2] implies the following corollary.

COROLLARY 2.4. *For each closed set* $C \subset [0,+\infty)$,

$$\limsup_{T\to+\infty}\limsup_{N\to+\infty}\frac{1}{NT}\log\mathbb{P}^N_{\nu_e^N}(q_{N,T}\in C)\le -\inf_{q\in C} i_e(q).$$

According to the arguments in [5], it should be possible to show that the same limiting function is obtained when the order of the limit in N and T is exchanged.

While we are not able to compute the limiting rate function $i_e(q)$ for $q>\bar q_e$, we are able to obtain an upper and a lower bound in terms of a "static strategy." Given $\pi\in\mathscr{P}_e$, with density f, set

$$R_2(\pi)=\frac12\int f f_* B\,\mathrm{d}\omega\,\mathrm{d}v\,\mathrm{d}v_*,\quad R_4(\pi)=\frac12\int\sqrt{f f_* f' f'_*}B\,\mathrm{d}\omega\,\mathrm{d}v\,\mathrm{d}v_*,$$

in which we denote by f, f_*, f', f'_* the density evaluated in v, v_*, v', v'_*. Note that, by Cauchy–Schwarz, $R_4(\pi)\le R_2(\pi)$.

THEOREM 2.5. *On* $[\bar q_e,+\infty)$,

$$i_e^-\le i_e\le i_e^+,$$

where

$$i_e^+(q)=\inf_{\pi\in\mathscr{P}_e}\left(q\log\frac{q}{R_4(\pi)}-q+R_2(\pi)\right)\tag{2.20}$$

and setting $\hat q_e=\sup_{\pi\in\mathscr{P}_e}R_4(\pi)>\bar q_e$,

$$i_e^-(q)=\begin{cases}0 & q\in[\bar q_e,\hat q_e]\\ q\log\dfrac{q}{\hat q_e}-q+\hat q_e & q\in(\hat q_e,+\infty).\end{cases}\tag{2.21}$$

In the proof, we show that the function i_e^+ corresponds to the optimal static strategy for the variational problem (2.16) obtained by restricting to paths not depending on time. It is possible that neither the upper nor the lower bound is sharp, which would correspond to the minimizer of the variational problem (2.16) exhibiting a nontrivial time dependence. We refer to [4, 6, 13] for examples of this phenomenon in other contexts.

The last topic that we discuss is the development by Γ-converge of the sequence of functions $\{\mathscr{I}_{e,T}(\cdot|m)\}_{T>0}$. In the terminology of large deviations, this corresponds to second order large deviation estimates. A similar phenomenon has been analyzed in the context of the so-called east model in [8, 9].

In contrast to the functional i_e describing the first order asymptotics, the functional describing the second order asymptotics still depends on the initial condition m. We limit the discussion to the particularly relevant case in which the initial velocities are sampled from the equilibrium probability. As before, we first introduce the limiting function. Recalling that M_ϵ, $\epsilon\in(0,+\infty)$ denotes the Maxwellian with zero average velocity and average energy ϵ, we let $j_e\colon[0,+\infty)\to[0,+\infty]$ be the lower semicontinuous function defined by

$$j_e(q):=\begin{cases}H_e\big(M_{\epsilon(q)}\big|M_e\big) & \text{if } q\in[0,\bar q_e],\\ +\infty & \text{if } q>\bar q_e,\end{cases}\tag{2.22}$$

where $\epsilon(q):=e(q/\bar q_e)^2$. By direct computation for $q\in[0,\bar q_e]$ we have

$$j_e(q)=\log\left(\frac{\bar q_e}{q}\right)^d.\tag{2.23}$$

THEOREM 2.6. *The sequence of functions* $\{\mathscr{I}_{e,T}(\cdot|M_e)\}_{T>0}$ *on* $[0,+\infty)$ *is equi-coercive and* Γ*-converges to* j_e *as* $T\to\infty$. *Namely,*

(i) *for each* $\ell>0$, *there is a compact* $K_\ell\subset\subset[0,+\infty)$ *such that* $\{q\in[0,+\infty)\colon\ \mathscr{I}_{e,T}(\cdot|m)\le\ell\}\subset K_\ell$ *eventually as* $T\to\infty$,

(ii) *for each* $q\in[0,+\infty)$ *and each sequence* $q_T\to q$, *the inequality* $\liminf_{T\to\infty}\mathscr{I}_{e,T}(q_T|M_e)\ge j_e(q)$ *holds, and*

(iii) *for each* $q\in[0,+\infty)$, *there exists a sequence* $q_T\to q$ *such that* $\limsup_{T\to\infty}\mathscr{I}_{e,T}(q_T|M_e)\le j_e(q)$.

In view of the standard property of Γ-convergence, see, e.g. [19], this statement together with the large deviations upper bound with fixed T in [2] implies that the sequence of real positive random variables $\{q_{N,T}\}_{T,N}$ satisfies a large deviations upper bound in the limit in which first $N\to\infty$ and then $T\to\infty$ with speed N and rate function $j_e(\cdot|m)$. In contrast to the case of the first order asymptotics described by Theorem 2.3, here the order of the limiting procedure does matter. In fact, as the large deviations speed does not depend on T, a large deviation principle in the limit in which first $T\to\infty$ and then $N\to\infty$ would be meaningless.

3. Reversibility. The next statement is the counterpart at the level of the rate functional of the reversibility of the microscopic dynamics. Let $\Upsilon\colon(\mathbb{R}^d)^4\to(\mathbb{R}^d)^4$ be the involution that exchanges the incoming and outgoing velocities, that is

$$\Upsilon(v,v_*,v',v'_*)=(v',v'_*,v,v_*). \tag{3.1}$$

Recalling the definitions of the functionals H_e and $J_{e,T}$ in (2.10), (2.12), we have the following identity.

PROPOSITION 3.1. *Fix* $T>0$. *For each* $(\boldsymbol{\pi},\boldsymbol{Q})\in\mathscr{S}_{e,T}$,

$$H_e(\boldsymbol{\pi}_0|M_e)+J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q})=H_e(\boldsymbol{\pi}_T|M_e)+J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q}\circ\Upsilon). \tag{3.2}$$

Moreover, if either side in (3.2) *is finite, then for all* $t\in[0,T]$, $\boldsymbol{\pi}_t$ *admits a density* f_t *enjoying the integrability*

$$\boldsymbol{Q}\left(\left|\log\frac{f'f'_\star}{ff_\star}\right|\right)<\infty. \tag{3.3}$$

Finally, for any $[r,s]\subset[0,T]$, (3.2) *the entropy satisfies the chain rule*

$$H_e(f_s\,\mathrm{d}v|M_e)-H_e(f_r\,\mathrm{d}v|M_e)=\boldsymbol{Q}\left(1_{[r,s]}\log\frac{f'f'_\star}{ff_\star}\right). \tag{3.4}$$

In particular, $t\mapsto H_e(\boldsymbol{\pi}_t|M_e)$ *is finite and continuous in time.*

We understand that identity (3.2) holds in the sense that if either side is finite, then also the other one is finite and equality holds. The strategy of the proof will be to show that any $(\boldsymbol{\pi},\boldsymbol{Q})$ for which the left-hand side $\mathcal{I}_{e,T}(\boldsymbol{\pi},\boldsymbol{Q})$ of (3.2) is finite can be approximated by regular trajectories, for which we can compute a version of (3.4) as follows:

$$H_e(f_sdv|M_e)-H_e(f_rdv|M_e)=2\int_r^s(Q_u^{(1)}(\log f)-Q_u^{(3)}(\log f))du,$$

where $\boldsymbol{Q}=\mathrm{d}t\,Q_t$ for any $[r,s]\subset[0,T]$, and where the superscript on the right-hand side denotes the marginals, namely,

$$Q_t^{(1)}(\cdot)=\int Q_t(\,\cdot\,,\mathrm{d}v_*,\mathrm{d}v',\mathrm{d}v'_*),\qquad Q_t^{(3)}(\cdot)=\int Q_t(\mathrm{d}v',\mathrm{d}v'_*,\,\cdot\,,\mathrm{d}v_*).$$

On such regular paths, the chain rule can be rearranged to find the identity (3.2), and the approximations are constructed so as to be able to pass to the limit. The finiteness of the right-hand side of (3.2) will then imply the claimed integrability (3.3), which allows us to pass to the limit to find (3.4).

Proof. We fix, for the duration of the proof, a pair $(\boldsymbol{\pi},\boldsymbol{Q})$ for which the left-hand side of (3.2) is finite; in the following steps, we will show that the right-hand side is finite, and

$$H_e(\boldsymbol{\pi}_T|M_e)+J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q}\circ\Upsilon)\le H_e(\boldsymbol{\pi}_0|M_e)+J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q}). \tag{3.5}$$

The converse inequality is then proven by applying the previous case to the time-reversed path

$$\hat{\boldsymbol{\pi}}_t:=\boldsymbol{\pi}_{T-t};\qquad \hat{\boldsymbol{Q}}=(\Upsilon\circ Q_{T-t})dt. \tag{3.6}$$

Step 1. Bounds. We begin with some estimates. In [2, 14], it is shown that $J_{e,T}$ admits the variational representation

$$J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q})=\sup_F(\boldsymbol{Q}(F)-\boldsymbol{Q}^\pi(\mathrm{e}^F-1))=:E(\boldsymbol{Q}|\boldsymbol{Q}^\pi),$$

where the supremum is carried out over all continuous and bounded $F\colon[0,T]\times(\mathbb{R}^d)^2\times(\mathbb{R}^d)^2\to\mathbb{R}$ such that $F(t;v,v_*,v',v'_*)=F(t;v_*,v,v',v'_*)=F(t;v,v_*,v'_*,v')$. From this, it immediately follows that $E(\cdot|\cdot)$ is convex and lower semicontinuous in both arguments.

Since $J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q})$ is finite, by choosing $F=\log(1+|v|+|v_*|)$ and using a standard truncation argument, we obtain that $\boldsymbol{Q}(\log(1+|v|+|v_*|))<+\infty$, and similarly $\boldsymbol{Q}(\log(1+|v'|+|v'_*|))<+\infty$. Moreover, by choosing $F=-\log(B)$, we obtain $\boldsymbol{Q}(\log 1/B)<+\infty$. Since $B=|\omega\cdot(v-v_*)|/2$, we conclude that $\boldsymbol{Q}(|\log B|)$ is finite.

Finally, under the additional assumption that π_t admits a strictly positive density $f_t>0$ for all $t\in[0,T]$, taking

$$F=\log\left(\frac{1}{(1+|v|)^\alpha(1+|v_*|)^\alpha f(v)f(v_*)}\right)$$

with $\alpha>d+1$ and recalling that $\boldsymbol{Q}(\log(1+|v|+|v_*|)$ is bounded, we conclude that $-\boldsymbol{Q}(\log f)<+\infty$. Moreover, if f is bounded, then $\boldsymbol{Q}(|\log f(v)|)<+\infty$.

Step 2. Velocity regularization. We first perform a regularization in the velocity variables.

Given a pair $(\boldsymbol{\pi},\boldsymbol{Q})$, we denote by $\boldsymbol{\pi}_t(\mathrm{d}v)=f_t\,\mathrm{d}v$ and $\mathrm{d}\boldsymbol{Q}=Q_t\,\mathrm{d}t\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega$. Given $0<\varepsilon<1$, let g_ε be the Gaussian kernel on $\mathbb{R}^d$ with variance ε and let $\alpha_\varepsilon\to 1$ be given by $\alpha_\varepsilon:=\sqrt{\frac{d}{2e}\varepsilon+1}$. For each $\varepsilon\in(0,1)$, we construct the new path $(\boldsymbol{\pi}^\varepsilon,\boldsymbol{Q}^\varepsilon)$ by taking the density of $\boldsymbol{\pi}^\varepsilon$ to be

$$f^\varepsilon(v)=\alpha_\varepsilon^d(g_\varepsilon*f)(\alpha_\varepsilon v)$$

and setting

$$\boldsymbol{Q}^\varepsilon=\alpha_\varepsilon^{2d}((g_\varepsilon\otimes g_\varepsilon\otimes\mathrm{id})*\boldsymbol{Q})(\alpha_\varepsilon v,\alpha_\varepsilon v^*,\omega).$$

As an immediate result of the definition, it holds that

$$\boldsymbol{\pi}_t^\varepsilon(\zeta_0)=\frac{1}{\alpha_\varepsilon^2}\left(\frac{d}{2}\varepsilon+\boldsymbol{\pi}_t(\zeta_0)\right)\le e$$

and that $(\boldsymbol{\pi}^\varepsilon, \boldsymbol{Q}^\varepsilon)$ satisfies the balance equation. Using the bounds in step 1, we can rewrite

$$J_{e,T}(\boldsymbol{\pi}, \boldsymbol{Q}) = E(\boldsymbol{Q}|\boldsymbol{P}^\pi) - \boldsymbol{Q}(\log B) + \boldsymbol{Q}^\pi(1) - \boldsymbol{P}^\pi(1), \tag{3.7}$$

where $\mathrm{d}\boldsymbol{P}^\pi = \mathrm{d}t\,\mathrm{d}\omega\pi_t(\mathrm{d}v)\pi_t(\mathrm{d}v_*)$. Since $\left(\boldsymbol{P}^\pi\right)^\varepsilon = \boldsymbol{P}^{\pi^\varepsilon}$, by convexity and lower semicontinuity, $E(\boldsymbol{Q}^\varepsilon|\boldsymbol{P}^{\pi^\varepsilon}) \to E(\boldsymbol{Q}|\boldsymbol{P}^\pi)$. Moreover, one can prove that $\boldsymbol{Q}^\varepsilon(|\log B|)$ is finite and

$$\lim_{\varepsilon\to 0} \boldsymbol{Q}^\varepsilon(\log B) = \boldsymbol{Q}(\log B), \qquad \lim_{\varepsilon\to 0} \boldsymbol{Q}^{\pi^\varepsilon}(1) = \boldsymbol{Q}^\pi(1). \tag{3.8}$$

Splitting $\log B$ into its positive and negative parts, the convergence of the negative part is guaranteed by the argument of [2, Appendix A]. In the positive part, the argument is different because we no longer (in contrast to the cited paper) assume that $\boldsymbol{Q}(\zeta_0) < \infty$; however, the argument may be completed by noticing that $(\log B)^+ \le \log(1+|v|+|v_\star|)$, which is guaranteed to be integrable thanks to the bounds established in step 1.

Moreover, $\boldsymbol{P}^{\pi^\varepsilon}(1) = \boldsymbol{P}(1)$. Therefore, $J_{e,T}(\boldsymbol{\pi}^\varepsilon, \boldsymbol{Q}^\varepsilon)$ is finite and converges to $J_{e,T}(\boldsymbol{\pi}, \boldsymbol{Q})$ as $\varepsilon \to 0$.

Step 3. Time regularization. In order to complete the approximation by smooth trajectories, we must also regularize in the time variable; it will be convenient to keep the parameters independent. Writing $(f_t^\varepsilon, Q_t^\varepsilon)$ for the velocity-regularized pair constructed in the previous step, let $\imath_\eta$ be a smooth approximation of the Dirac measure in $\mathbb{R}$, with support in $(0,\eta)$, and set

$$(\tilde f_t^\varepsilon, \tilde Q_t^\varepsilon) = \begin{cases} (f_r^\varepsilon, 0) & t \le 0 \\ (f_t^\varepsilon, Q_t^\varepsilon) & t > 0. \end{cases}$$

We now define $(\boldsymbol{\pi}^{\eta,\varepsilon}, \boldsymbol{Q}^{\eta,\varepsilon})$ to be the path with densities $f_t^{\eta,\varepsilon} = (\imath_\eta * \tilde f^\varepsilon)_t$ and $Q_t^{\eta,\varepsilon} = (\imath_\eta * \tilde Q^\varepsilon)_t$, where $\imath_\eta *$ is the convolution in time. Observe that $(\boldsymbol{\pi}^{\eta,\varepsilon}, \boldsymbol{Q}^{\eta,\varepsilon})$ satisfies the balance equation.

Let $J_{[0,s]}$ be the functional $J_{e,T}$ when the interval $[0,T]$ is replaced by $[0,s]$, and we have dropped the dependence on e. From now on, we can follow [2, Theorem 5.6, step 3], obtaining that for each $\varepsilon > 0$,

$$\lim_{\eta\to 0} J_{[0,s]}(\boldsymbol{\pi}^{\eta,\varepsilon}, \boldsymbol{Q}^{\eta,\varepsilon}) = J_{[0,s]}(\boldsymbol{\pi}^\varepsilon, \boldsymbol{Q}^\varepsilon).$$

Step 4. Entropy chain rule for regular paths. By construction, $f^{\eta,\varepsilon}$ is regular in (t,v) and strictly positive. The same holds for all the marginal densities $Q_t^{\eta,\varepsilon,(i)}$, $i = 1,3$, defined as

$$Q_t^{\eta,\varepsilon,(1)}(v) = \int \mathrm{d}v_*\,\mathrm{d}\omega\, Q_t(v, v_*, \omega)$$
$$Q_t^{\eta,\varepsilon,(3)}(v) = \int \mathrm{d}v_*\,\mathrm{d}\omega\, Q_t(v', v'_*, \omega),$$

where we recall the notation $Q_t^{\eta,\varepsilon,(2)} = Q_t^{\eta,\varepsilon,(1)}$ and $Q_t^{\eta,\varepsilon,(4)} = Q_t^{\eta,\varepsilon,(3)}$. As a consequence, we can write the balance equation pointwise as

$$\partial_t f_t^{\eta,\varepsilon} = 2(Q_t^{\eta,\varepsilon,(3)} - Q_t^{\eta,\varepsilon,(1)}).$$

Since $\boldsymbol{\pi}^{\eta,\varepsilon}$ admits bounded densities $f^{\eta,\varepsilon}(t,v)$, the final bound in step 1 applies to prove

$$\boldsymbol{Q}^{\eta,\varepsilon}(|\log f^{\eta,\varepsilon}(v)|+|\log f^{\eta,\varepsilon}(v')|)=\boldsymbol{Q}^{\eta,\varepsilon,(1)}\big(|\log f^{\eta,\varepsilon}|\big)+\boldsymbol{Q}^{\eta,\varepsilon,(3)}\big(|\log f^{\eta,\varepsilon}|\big)<+\infty. \tag{3.9}$$

Therefore, using that $E(\boldsymbol{Q}^{\eta,\varepsilon}|\boldsymbol{P}^{\boldsymbol{\pi}^{\eta,\varepsilon}})<+\infty$, as follows from (3.7) and $\boldsymbol{Q}^{\eta,\varepsilon}(|\log B|)<+\infty$, we get that $\boldsymbol{Q}^{\eta,\varepsilon}\big(|\log Q^{\eta,\varepsilon}|\big)$ is finite. For any $t\in[0,T]$,

$$\partial_t(f_t^{\eta,\varepsilon}\log f_t^{\eta,\varepsilon})=2Q_t^{\eta,\varepsilon,(3)}(1+\log f_t^{\eta,\varepsilon})-2Q_t^{\eta,\varepsilon,(1)}(1+\log f_t^{\eta,\varepsilon}). \tag{3.10}$$

Since the reference measure is the Maxwellian M_e, we may use the representation (2.10) of $H_e(\cdot|M_e)$, so that the problem reduces to studying the evolution of $\int f_t^{\eta,\epsilon}\log f_t^{\eta,\epsilon}dv$. At time $t=0$, $H_e(f_0^{\eta,\varepsilon}\,\mathrm{d}v|M_e)=H_e(f_0^{0,\varepsilon}\,\mathrm{d}v|M_e)$ is finite. Integrating in $t\in[0,s]$ and in $v\in\mathbb{R}^d$, we conclude that

$$H_e(f_s^{\eta,\varepsilon}\,\mathrm{d}v|M_e)-H_e(f_0^{\varepsilon}\,\mathrm{d}v|M_e)=2\boldsymbol{Q}_{[0,s]}^{\eta,\varepsilon}(\log f^{\eta,\varepsilon}(v'))-2\boldsymbol{Q}_{[0,s]}^{\eta,\varepsilon}(\log f^{\eta,\varepsilon}(v)), \tag{3.11}$$

where $\boldsymbol{Q}_{[0,s]}$ is the restriction of $\boldsymbol{Q}$ to the time window $[0,s]$.

Step 5. Computation on regularized paths. We now perform a computation, still at the level of the regularized paths, in order to link the integrals appearing on the right-hand side of (3.11) to the rate function $J_{[0,s]}$ appearing in the desired conclusion (3.2); we refer to [15, Section 6.5.2] for a similar argument. Since $f^{\eta,\varepsilon}$ is everywhere positive, one can check that $\mathbf{Q}^{\pi^{\eta,\varepsilon}}\circ\Upsilon$ is absolutely continuous with respect to $\mathbf{Q}^{\pi^{\eta,\varepsilon}}$ with a density given by

$$\frac{\mathrm{d}(\mathbf{Q}^{\pi^{\eta,\varepsilon}}\circ\Upsilon)}{\mathrm{d}\mathbf{Q}^{\pi^{\eta,\varepsilon}}}=\frac{f_t^{\eta,\varepsilon}(v')f_t^{\eta,\varepsilon}(v'_\star)}{f_t^{\eta,\varepsilon}(v)f_t^{\eta,\varepsilon}(v_\star)}.$$

Meanwhile, since Υ is an involution, and the finiteness of $J_{e,T}(\boldsymbol{\pi}^{\eta,\varepsilon},Q^{\eta,\varepsilon})$ implies that $\boldsymbol{Q}^{\eta,\varepsilon}$ is absolutely continuous with respect to $\boldsymbol{Q}^{\pi^{\eta,\varepsilon}}$, it follows that $\mathbf{Q}^{\eta,\varepsilon}\circ\Upsilon$ has a density with respect to $\mathbf{Q}^{\pi^{\eta,\varepsilon}}\circ\Upsilon$ given by $\frac{\mathrm{d}\mathbf{Q}^{\eta,\varepsilon}}{\mathrm{d}\mathbf{Q}^{\pi^{\eta,\varepsilon}}}\circ\Upsilon$. By the chain rule for Radon–Nidokym derivatives, we finally see that $\mathbf{Q}^{\eta,\varepsilon}\circ\Upsilon$ is absolutely continuous with respect to $\boldsymbol{Q}^{\pi^{\eta,\varepsilon}}$ and

$$\frac{\mathrm{d}(\mathbf{Q}^{\eta,\varepsilon}\circ\Upsilon)}{\mathrm{d}\mathbf{Q}^{\pi^{\eta,\varepsilon}}}=\frac{f_t^{\eta,\varepsilon}(v')f_t^{\eta,\varepsilon}(v'_\star)}{f_t^{\eta,\varepsilon}(v)f_t^{\eta,\varepsilon}(v_\star)}\left(\frac{\mathrm{d}\mathbf{Q}^{\eta,\varepsilon}}{\mathrm{d}\mathbf{Q}^{\pi^{\eta,\varepsilon}}}\circ\Upsilon\right).$$

Substituting into the definition (2.12) of J and using the additivity of the logarithm, it follows that

$$J_{[0,s]}(\boldsymbol{\pi}^{\eta,\varepsilon},\boldsymbol{Q}^{\eta,\varepsilon}\circ\Upsilon)=J_{[0,s]}(\boldsymbol{\pi}^{\eta,\varepsilon},\boldsymbol{Q}^{\eta,\varepsilon})+(\mathbf{Q}_{[0,s]}^{\eta,\varepsilon}\circ\Upsilon)\left(\log\frac{f^{\eta,\varepsilon}(v')f^{\eta,\varepsilon}(v'_\star)}{f^{\eta,\varepsilon}(v)f^{\eta,\varepsilon}(v_\star)}\right). \tag{3.12}$$

Thanks to (3.9), the definition of Υ, and the symmetry in $(v,v')\leftrightarrow(v_\star,v'_\star)$, we may further break up the last term as

$$(\mathbf{Q}_{[0,s]}^{\eta,\varepsilon}\circ\Upsilon)\left(\log\frac{f^{\eta,\varepsilon}(v')f^{\eta,\varepsilon}(v'_\star)}{f^{\eta,\varepsilon}(v)f^{\eta,\varepsilon}(v_\star)}\right)=2\mathbf{Q}_{[0,s]}^{\eta,\varepsilon}(\log f^{\eta,\varepsilon}(v))-2\mathbf{Q}_{[0,s]}^{\eta,\varepsilon}(\log f^{\eta,\varepsilon}(v')). \tag{3.13}$$

Together with (3.11), we obtain the balance equation for the entropy on the regularized paths

$$H_e(f_s^{\eta,\varepsilon}\,\mathrm{d}v|M_e)+J_{[0,s]}(\boldsymbol{\pi}^{\eta,\varepsilon},\boldsymbol{Q}^{\eta,\varepsilon}\circ\Upsilon)=H_e(f_0^{\varepsilon}\,\mathrm{d}v|M_e)+J_{[0,s]}(\boldsymbol{\pi}^{\eta,\varepsilon},\boldsymbol{Q}^{\eta,\varepsilon}). \tag{3.14}$$

Step 6. Proof of (3.2). We now pass to the limit $\eta \to 0$ in (3.14). The right-hand side converges by step 3. By lower semicontinuity,

$$H_e(f_s^\varepsilon \,\mathrm{d}v|M_e) + J_{[0,s]}(\boldsymbol{\pi}^\varepsilon, \boldsymbol{Q}^\varepsilon \circ \Upsilon) \le H_e(f_0^\varepsilon \,\mathrm{d}v|M_e) + J_{[0,s]}(\boldsymbol{\pi}^\varepsilon, \boldsymbol{Q}^\varepsilon).$$

Now we pass to the limit $\varepsilon \to 0$. By (2.10), $H_e(\cdot|M_e)$ is convex. Then, by Jensen's inequality, step 2, and the lower semicontinuity, we deduce the more general version of (3.5) for any time interval $[0,s]$:

$$H_e(\pi_s|M_e) + J_{[0,s]}(\boldsymbol{\pi}, \boldsymbol{Q} \circ \Upsilon) \le H_e(\pi_0|M_e) + J_{[0,s]}(\boldsymbol{\pi}, \boldsymbol{Q}). \tag{3.15}$$

This extends to any interval $[r,s] \subset [0,T]$ by taking differences, and the special case $s = T$ yields (3.5). The same argument applied to the time-reversed path (3.6) shows that the previous inequality is actually an equality, and so is the version applied to a subinterval $[r,s]$

$$H_e(\pi_s|M_e) + J_{[r,s]}(\boldsymbol{\pi}, \boldsymbol{Q} \circ \Upsilon) = H_e(\pi_r|M_e) + J_{[r,s]}(\boldsymbol{\pi}, \boldsymbol{Q}), \tag{3.16}$$

generalizing the claim (3.2).

Step 7. Density, integrability and chain rule. Let us now specialize to the case where the left-hand side, and hence both sides, of (3.2) are finite. As a result, the right-hand side of (3.15) is bounded in $s \in [0,T]$, and since J is nonnegative, it follows that $\sup_{s\le T} H_e(\pi_s|M_e) < \infty$, from which it follows that every π_s admits a density $\pi_s = f_s dv$. Using the finiteness of $J_{[0,T]}(\boldsymbol{\pi}, \boldsymbol{Q} \circ \Upsilon)$, the same argument as in step 5 identifies

$$\frac{\mathrm{d}(\boldsymbol{Q} \circ \Upsilon)}{\mathrm{d}\boldsymbol{Q}^\pi} = \frac{f_t(v')f_t(v'_\star)}{f_t(v)f_t(v_\star)} \left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}\boldsymbol{Q}^\pi} \circ \Upsilon\right). \tag{3.17}$$

In particular, the first factor is finite $\boldsymbol{Q}^\pi$-almost everywhere. As a result, it follows that

$$\left(\log \frac{f' f'_\star}{f f_\star}\right)_+ \le \left(\log \frac{\mathrm{d}(\boldsymbol{Q} \circ \Upsilon)}{\mathrm{d}\boldsymbol{Q}^\pi}\right)_+ + \left(\log \frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}\boldsymbol{Q}^\pi} \circ \Upsilon\right)_-$$

with $\pm$ denoting the positive, respectively negative, parts of a function and $_\star,'$ indicating the arguments at which f_t is evaluated. We now integrate both sides with respect to $\boldsymbol{Q} \circ \Upsilon$ and use the finiteness of

$$J_{[0,T]}(\boldsymbol{\pi}, \boldsymbol{Q} \circ \Upsilon) = (\boldsymbol{Q} \circ \Upsilon)\left(\log \frac{\mathrm{d}(\boldsymbol{Q} \circ \Upsilon)}{\mathrm{d}\boldsymbol{Q}^\pi}\right) - (\boldsymbol{Q} \circ \Upsilon)(1) + \boldsymbol{Q}^\pi(1)$$

to get

$$\begin{aligned}\boldsymbol{Q}\left(\left(\log \frac{f f_\star}{f' f'_\star}\right)_+\right) &\le J_{[0,T]}(\boldsymbol{\pi}, \boldsymbol{Q} \circ \Upsilon) + (\boldsymbol{Q} \circ \Upsilon)(1) \\ &\quad + (\boldsymbol{Q} \circ \Upsilon)\left(\left(\log \frac{\mathrm{d}(\boldsymbol{Q} \circ \Upsilon)}{\mathrm{d}\boldsymbol{Q}^\pi}\right)_-\right) + \boldsymbol{Q}\left(\left(\log \frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}\boldsymbol{Q}^\pi}\right)_-\right).\end{aligned}$$

The last two terms are readily seen to be finite using the boundedness of $k(\log k)_-$ and the finiteness of $\boldsymbol{Q}^\pi(1)$, and we ultimately conclude that

$$\boldsymbol{Q}\left(\left(\log \frac{f f_\star}{f' f'_\star}\right)_+\right) < \infty.$$

The argument for the negative part is similar, and we conclude the claimed integrability (3.3).

The deduction of (3.4) from (3.16) proceeds as in step 5, and the continuity of $t \mapsto H_e(\boldsymbol{\pi}_t|M_e)$ follows by dominated convergence. ∎

4. Controllability of the Boltzmann equation.

The following result, which will be used in deriving Proposition 2.2 and Theorem 2.3, shows that any two probability measures may be joined by a path of finite cost.

THEOREM 4.1. *Fix $T > 0$ and $e > 0$. There exists $C_1 = C_1(e)$ and a function $F_1 : [0,\infty)^3 \to [0,\infty)$ for which the following holds. Given $\pi_i \in \mathscr{P}_e$ with bounded entropy, $i = 1,2$ and $\kappa \geq C_1(e)$, there exists a path $(\boldsymbol{\pi},\boldsymbol{Q}) \in \mathcal{S}_{e,T}$ such that $\boldsymbol{\pi}_0 = \pi_1$, $\boldsymbol{\pi}_T = \pi_2$, and*

$$J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q}) \leq F_1(e,T,\kappa) + H_e(\pi_2|M_e), \qquad \boldsymbol{Q}(1) = \kappa.$$

The function F may be chosen such that, for fixed e,κ, for all $0 < T_0 < T_1 < \infty$, it holds that

$$\sup_{T_0 \leq T \leq T_1} F_1(e,T,\kappa) < \infty. \tag{4.1}$$

In order to contextualize this result, which may have independent interest, we first discuss an alternative formulation of the dynamical rate function $J_{e,T}$ in (2.12) in terms of a control problem for the homogeneous Boltzmann equation (2.6); see also [17, 14].

On a fixed time interval $[0,T]$, we define a cost functional

$$\widehat{J}_{e,T}(\boldsymbol{\pi}) := \frac{1}{2} \inf \left\{ \int_0^T \mathrm{d}t \int \boldsymbol{\pi}_t(dv)\boldsymbol{\pi}_t(dv_*)\,\mathrm{d}\omega\, B(v-v_*,\omega)\Psi(\boldsymbol{F}) \right\}. \tag{4.2}$$

where $\Psi\colon \mathbb{R} \to \mathbb{R}_+$ is defined by $\Psi(x) := xe^x - e^x + 1$, and the infimum runs over all controls $\boldsymbol{F}\colon [0,T] \times (\mathbb{R}^d)^4 \to \mathbb{R}$ for which the density f_t of π_t is a weak solution to the *controlled Boltzmann equation*

$$\begin{cases} \partial_t f_t(v) = \displaystyle\int \mathrm{d}v_*\,\mathrm{d}\omega\, B(v-v_*,\omega) e^{\boldsymbol{F}_t(v',v'_*,v,v_*)} f_t(v') f_t(v'_*) \\ \qquad\qquad - \displaystyle\int \mathrm{d}v_*\,\mathrm{d}\omega\, B(v-v_*,\omega) e^{\boldsymbol{F}_t(v,v_*,v',v'_*)} f_t(v) f_t(v_*), \quad (t,v) \in (0,T) \times \mathbb{R}^d \\ f_0\,\mathrm{d}v = \pi_0, \end{cases} \tag{4.3}$$

with the energy never exceeding e. Note that Ψ is positive with a unique quadratic minimum achieved at $x = 0$. Observe also that the case $\boldsymbol{F} = -\infty$, for which $\widehat{J}_{e,T}(\boldsymbol{F}) < +\infty$, corresponds to a vanishing perturbed collision kernel. With this definition, it holds for all $\boldsymbol{\pi}$ that

$$\widehat{J}_{e,T}(\boldsymbol{\pi}) = \inf_{\boldsymbol{Q}} J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q}). \tag{4.4}$$

In this way, Theorem 4.1 may be understood as asserting that, given $\pi_1,\pi_2 \in \mathscr{P}_e$ with bounded entropy, there exists a control $\boldsymbol{F}$ and a solution $\boldsymbol{\pi}$ to (4.3) such that $\boldsymbol{\pi}_0 = \pi_1$, $\boldsymbol{\pi}_T = \pi_2$, and such that the integral appearing in (4.2) is finite.

Let us remark that arguments from (4.2) run into issues of nonuniqueness, even for the (uncontrolled) Boltzmann equation (2.6). Given an initial datum f_0, taking

$F = 0$ reduces (4.3) to the uncontrolled Boltzmann equation (2.6), for which there are multiple solutions [18]. For this reason, we will not make precise the notion of admissible controls and have rather formulated Theorem 4.1 in terms of $J_{e,T}$. We emphasize that we do not need the initial and target measures to have the same energy.

Strategy of the proof. Before proving the full statement of Theorem 4.1, we first prove a "one-sided" version in Lemma 4.2, which is the special case where π_2 is a Maxwellian M_e. In this case, we can construct the path $\boldsymbol{\pi}$ by taking a solution to the homogeneous Boltzmann equation starting at π_1 and which has energy e for all times $t > 0$, and then reparameterizing time. Such a solution always exists as either the unique energy-conserving solution if π_1 already has energy e or a Lu–Wennberg solution otherwise. The total collision rate will be tuned to a desired value κ by using two different flux measures Q^1_τ, Q^2_τ for the nonreparametrized solution: Q^1_τ will be the "natural" measure associated with the solution, while Q^2_τ will remove certain collisions. By switching over after a suitably chosen time, we can guarantee that the total number of collisions reaches a desired finite value κ, which will also ensure that the rate remains finite after the change in time-scale. The "two-sided" case is argued separately and is given by concatenating the path $\pi_1 \to M_e$ produced by Lemma 4.2 with the time-reversal of the path $\pi_2 \to M_e$ using Proposition 3.1.

LEMMA 4.2. *For every $e, T > 0$, there exist $C_2 = C_2(e)$ and a function $F_2 : [0,\infty)^3 \to [0,\infty)$ satisfying* (4.1) *such that for every $\pi \in \mathscr{P}_e$ and $\kappa \geq C_1(e)$, there exists $(\boldsymbol{\pi}, \boldsymbol{Q}) \in \mathcal{S}_{e,T}$ such that $\boldsymbol{\pi}_0 = \pi$, $\boldsymbol{\pi}_T = M_e$, and*

$$J_{e,T}(\boldsymbol{\pi}, \boldsymbol{Q}) \leq F_2(e, T, \kappa), \qquad \boldsymbol{Q}(1) = \kappa.$$

Proof. We divide into steps; fix everywhere e, T and π as in the lemma. A nonreparametrized solution is defined in step 1, and the reparameterization is given in step 2, yielding the final $(\boldsymbol{\pi}, \boldsymbol{Q})$. The asserted bounds are proven in steps 3–5.

Step 1. Infinite-time path. Let us take $\left(f_\tau\right)_{\tau\geq 0}$ to be *any* solution to the homogeneous Boltzmann equation whose initial datum f_0 is the density of π and whose associated measures $\pi_\tau = f_\tau dv$ satisfy $\pi_\tau(\zeta_0) = e$ for all $\tau > 0$. In the case when π already has energy e, then f_τ is the unique energy conserving solution, while if $\pi(\zeta_0) < e$, then it is a Lu–Wennberg solution with a single jump of the energy at $\tau = 0$. In view of [2, Lemma 7.6], such a solution exists and setting

$$Q^1_\tau := Q^{f_\tau}(\mathrm{d}v, \mathrm{d}v_*, \mathrm{d}\omega) := \frac{1}{2} f_\tau(v) f_\tau(v_*) B(v - v_*, \omega)\, \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\omega$$

produces paths in $\mathcal{S}_{e,T}$ with vanishing dynamical cost for any finite time horizon T. Moreover, since π_t is energy conserving away from $t = 0$, it also satisfies any bounds valid for the energy conserving equation (e.g. [21]) which only require finite energy, replacing the initial energy by e if necessary.

In the sequel, we will use a different flow measure associated with the solution $(f_\tau)_{\tau\geq 0}$ in addition to Q^1_t given above.

$$Q^2_\tau(\mathrm{d}v, \mathrm{d}v_*, \mathrm{d}\omega) = \left[f_\tau(v) f_\tau(v_*) - f_\tau(v') f_\tau(v'_*)\right]_+ B(v - v_*, \omega)\, \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\omega, \tag{4.5}$$

in which $[\,\cdot\,]_+$ denotes the positive part.

Step 2. Time reparameterization. We now construct a path with a specified number of collisions on a finite time interval $[0, T]$. First, we observe that, for any $\tau > 0$, $Q^1_\tau(1) \geq Q^2_\tau(1)$. Moreover, a straightforward argument shows that $Q^1_\tau(1)$ is bounded below, uniformly in τ,π, so $\int_0^\infty Q_\tau^1(1)d\tau=\infty$. Let us define $\kappa_\star(\pi):=\int_0^\infty Q_\tau^2(1)d\tau$; we will see in step 3 below that there exists $C_2(e)<\infty$ such that $\kappa_\star(\pi)\le C_2(e)$ for any $\pi\in\mathscr{P}_e$. For any $\kappa\in[C_2(e),\infty)$, there is thus a unique $\tau_\star\in[0,\infty)$ such that

$$\int_0^{\tau_\star}(Q_\sigma^1-Q_\sigma^2)(1)d\sigma=\kappa-\kappa_\star(\pi)$$

and set

$$Q_\tau:=\begin{cases}Q_\tau^1, & \text{if } 0\le\tau<\tau_\star;\\ Q_\tau^2, & \text{otherwise}\end{cases} \tag{4.6}$$

we find that $\int_0^\infty Q_\tau(1)d\tau=\kappa$. We note, for future reference, that there exists $c(e)<\infty$ such that $Q_\tau^1(1)\ge c(e)^{-1}$ for all τ, from which it follows that

$$\tau_\star\le c(e)\kappa. \tag{4.7}$$

Setting $\phi\colon[0,T)\to[0,+\infty)$ to be $\phi(t)=(T-t)^{-1}-T^{-1}$, we define a time-reparametrized path $(\boldsymbol{\pi},\boldsymbol{Q})$ by

$$\boldsymbol{\pi}_t=f_{\phi(t)}\,\mathrm{d}v,\qquad \boldsymbol{Q}(\mathrm{d}t)=Q_{\phi(t)}\phi'(t)\,\mathrm{d}t, \tag{4.8}$$

where we understand that $\boldsymbol{\pi}_T=M_e$. In the remaining steps, we establish an upper bound $\kappa_\star(\pi)\le C_1(e)$, which allows the previous construction for any $\kappa\ge C_1(e)$ that $(\boldsymbol{\pi},\boldsymbol{Q})\in\mathcal{S}_{e,T}$ and the claimed bound on $J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q})$.

Step 3. Bound on $\boldsymbol{\kappa_\star(\pi)}$. In this step, we prove that there exists $C_2=C_2(e)$ depending only on the energy e, such that $\kappa_\star(\pi)\le C_2(e)$ for all $\pi\in\mathscr{P}_e$. We denote by $\|\cdot\|_{\mathrm{TV}}$ the total variation norm on the space of signed measures. By [21], there exist $\gamma=\gamma(e)>0$ and $C=C(e)<+\infty$ such that for any f_0 with energy e,

$$\big\|f_\tau\,\mathrm{d}v-M_e\big\|_{\mathrm{TV}}\le Ce^{-\gamma\tau},\qquad \tau\ge0. \tag{4.9}$$

Writing g for the density of M_e, recalling (4.5), and using that $g(v)g(v_*)B(v-v_*,\omega)$ is symmetric with respect to Υ, we have

$$\begin{aligned}\big\|Q^{f_\tau}-Q^{M_e}\big\|_{\mathrm{TV}}&\le\int\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\big|f_\tau(v)f_\tau(v_*)-g(v)g(v_*)\big|B(v-v_*,\omega)\\&\le2\int\mathrm{d}v\,\mathrm{d}v_*\,|v|\,\big|f_\tau(v)f_\tau(v_*)-g(v)g(v_*)\big|\\&\le2\big\|f_\tau\,\mathrm{d}v-M_e\big\|_{\mathrm{TV}}\int\mathrm{d}v\,|v|f_\tau(v)+2\int\mathrm{d}v\,|v|\big|f_\tau(v)-g(v)\big|.\end{aligned}$$

Since f_τ has energy e for any $\tau>0$, the first term can be directly bounded by using (4.9). In order to bound the second, given $\ell>0$ we have

$$\begin{aligned}&\int\mathrm{d}v\,|v|\big|f_\tau(v)-g(v)\big|\\&\quad\le\ell\int\mathrm{d}v\,\big|f_\tau(v)-g(v)\big|+\int_{|v|\ge\ell}\mathrm{d}v\,|v|\,\big|f_\tau(v)-g(v)\big|\le\ell Ce^{-\gamma\tau}+4\,\frac{e}{\ell},\end{aligned}$$

where we used (4.9) and the Chebyshev inequality, noticing that both f_τ and g have energy e. Optimizing at $\ell=e^{\gamma\tau/2}$, we find that, for some $\gamma'=\gamma'(e)>0$ and $C'=C'(e)<+\infty$,

$$\big\|Q^{f_\tau}-Q^{M_e}\big\|_{\mathrm{TV}}\le C'e^{-\gamma'\tau},\qquad \tau\ge0. \tag{4.10}$$

Recalling the definition (4.5) of Q^2_τ, the bound (4.10) implies

$$Q^2_\tau(1) \le 2C' e^{-\gamma'\tau}. \tag{4.11}$$

Recalling that C', γ' depend only on e and not on $\pi \in \mathscr{P}_e$, the claim thus holds when we define $C_2(e) := 2C'(e)/\gamma'(e)$ since $\kappa_\star(\pi)$ is defined to be the integral of the left-hand side over $t \in [0,\infty)$.

Step 4. Balance equation. We now check that (2.8) holds, first on the interval $[0,T)$ and then extending to the endpoint by continuity. First, by definition of $(f_\tau)_{\tau>0}$ and Q_τ, $\big((f_\tau)_{\tau\ge 0}\,\mathrm{d}v, (Q_\tau)_{\tau\ge 0}\,\mathrm{d}\tau\big)$ satisfies the balance equation (2.8) on $[0,\tau_\star]$ since on this interval $Q_\tau = Q^1_\tau$ and because f_τ satisfies the Boltzmann equation (2.6). For $\tau \ge \tau_\star$, observe, recalling (3.1),

$$Q^{f_\tau} = Q_\tau + \frac{1}{2}\big(Q^{f_\tau} + Q^{f_\tau}\circ \Upsilon\big) - \frac{1}{2}\big|Q^{f_\tau} - Q^{f_\tau}\circ\Upsilon\big|. \tag{4.12}$$

Since the last two terms on the right-hand side above are symmetric with respect to Υ,

$$\frac{\partial f_\tau}{\partial \tau}\,\mathrm{d}v = 2\big[\big(Q^{f_\tau}\big)^{(3)}(\mathrm{d}v) - \big(Q^{f_\tau}\big)^{(1)}(\mathrm{d}v)\big] = 2\big[\big(Q_\tau\big)^{(3)}(\mathrm{d}v) - \big(Q_\tau\big)^{(1)}(\mathrm{d}v)\big],$$

where the superscripts denote the marginals on the first and third variable, respectively, and hence the pair $\big((f_\tau)_{\tau\ge 0}\,\mathrm{d}v, (Q_\tau)_{\tau\ge 0}\,\mathrm{d}\tau\big)$ also satisfies the balance equation (2.8) on $\tau \ge \tau_\star$ and globally. Therefore, by changing the time variable, the pair $(\boldsymbol{\pi}, \boldsymbol{Q})$ defined in (4.8) also satisfies the balance equation on $[0,T)$ in that (2.8) holds when the terminal time T is replaced by any $t \in [0,T)$. In order to extend this to the endpoint T, we observe that

$$\int_t^T \boldsymbol{Q}(ds,1) \le 2C' \int_t^T (T-s)^{-2} \exp\big(-\gamma'((T-s)^{-1} - T^{-1})\big)\, ds \to 0$$

as $t \uparrow T$. In particular, we may pass to the limit of both sides when we evaluate the balance equation at $t < T$ and send $t \to T$. It follows that $(\boldsymbol{\pi}, \boldsymbol{Q})$ satisfies the balance equation on $[0,T]$ and that $(\boldsymbol{\pi}, \boldsymbol{Q}) \in \mathscr{S}_{e,T}$.

Step 5. Estimate of dynamic cost. To estimate $J_{e,T}(\boldsymbol{\pi}, \boldsymbol{Q})$, we notice that (4.12) implies $Q_\tau \le Q^{f_\tau}$, independently of whether or not $\tau \le \tau_\star$. We thus have

$$\begin{aligned} J_{e,T}(\boldsymbol{\pi}, \boldsymbol{Q}) &= \int_0^T \mathrm{d}t\,\phi'(t) \int \mathrm{d}Q_{\phi(t)} \log\Big(\phi'(t) \frac{dQ_{\phi(t)}}{dQ^{f_{\phi(t)}}}\Big) - \boldsymbol{Q}(1) + \int_0^T \mathrm{d}t\, Q^{f_{\phi(t)}}(1) \\ &\le \int_0^\infty \mathrm{d}\tau\, Q_\tau(1)\, \log\big(T^{-1} + \tau\big)^2 + T \sup_{\mu\in\mathscr{P}_e} \frac{1}{2} \int \mu(\mathrm{d}v)\mu(\mathrm{d}v_*)\,\mathrm{d}\omega\, B. \end{aligned}$$

The final term may be estimated by $c(2+4e)T^{-2}$ using a simple upper bound $B \le \frac{1}{2}(1 + |v - v_\star|^2)$, which is of the form required for (4.1). In the first integral, the contribution from $\tau \le \tau_\star$ is bounded by recalling that $\tau_\star(\pi) \le c(e)\kappa$ and that $Q_\tau(1) \le (2+4e)$, yielding a bound $(4+8e)c(e)\kappa \log(T^{-1} + c(e)\kappa)$, while the contribution from $\tau > \tau_\star$ is bounded by using (4.11). All of these bounds depend only on e, T, κ and have the property (4.1) asserted in the Lemma, so the proof is complete. ∎

Proof of Theorem 4.1. Fix e. Let us define $C_1(e) := 2C_2(e)$, where $C_2(e)$ is given by Lemma 4.2. For any $\kappa \ge C_1(e)$, let $(\boldsymbol{\pi}^i, \boldsymbol{Q}^i) \in \mathscr{S}_{e,T/2}$, $i = 1,2$, be the paths provided by Lemma 4.2, with $\pi = \pi^i$, $i = 1,2$ and T replaced by $T/2$ and κ replaced by $\frac{\kappa}{2}$. Denote by $\chi\colon [0,T/2]\to[0,T/2]$ be the time reflection $\chi(t)=T/2-t$, and let $\big(\hat{\boldsymbol{\pi}}^2,\hat{\boldsymbol{Q}}^2\big)$ be the path defined by

$$\hat{\boldsymbol{\pi}}^2_t := \boldsymbol{\pi}^2_{\chi(t)}, \qquad \hat{\boldsymbol{Q}}^2 := \boldsymbol{Q}^2\circ\chi\circ\Upsilon.$$

By direct computation, it satisfies the balance equation (2.8) so that $\big(\hat{\boldsymbol{\pi}}^2,\hat{\boldsymbol{Q}}^2\big)\in\mathscr{S}_{e,T/2}$. Finally, let $(\boldsymbol{\pi},\boldsymbol{Q})\in\mathscr{S}_{e,T}$ be the path obtained by concatenating $\big(\boldsymbol{\pi}^1,\boldsymbol{Q}^1\big)$ with $\big(\hat{\boldsymbol{\pi}}^2,\hat{\boldsymbol{Q}}^2\big)$. By construction, $\boldsymbol{\pi}_0=\pi_1$, $\boldsymbol{\pi}_T=\pi_2$, $\boldsymbol{Q}(1)=\boldsymbol{Q}^1(1)+\boldsymbol{Q}^2(1)=\kappa$, and thanks to Proposition 3.1,

$$\begin{aligned}J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q}) &= J_{e,T/2}(\boldsymbol{\pi}^1,\boldsymbol{Q}^1)+J_{e,T/2}(\boldsymbol{\pi}^2,\boldsymbol{Q}^2)+H_e(\pi_2|M_e)\\ &\le 2F_2(e,T/2,\kappa/2)+H_e(\pi_2|M_e),\end{aligned}$$

where F_2 is the function given by Lemma 4.2. This completes the proof with $F_1(e,T,\kappa)$ $:=2F_2(e,T/2,\kappa/2)$. ∎

5. First order asymptotics. We collect the proofs of the main results related to the first order asymptotic.

Proof of Proposition 2.2.

Step 1. The identity (2.19). Since all paths $(\boldsymbol{\pi},\boldsymbol{Q})\in\mathscr{A}_{e,T}(q|\mu)$ start and end at $\pi_0=\pi_T=\mu$, one may concatenate competitors and use the translation covariance of $J_{e,T}$ to find that the function

$$i_e(q|\mu,T) := \inf_{(\boldsymbol{\pi},\boldsymbol{Q})\in\mathcal{A}_{e,T}(q|\mu)} J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q}) \tag{5.1}$$

enjoys the subadditivity property

$$i_e(q|\mu,T_1+T_2)\le i_e(q|\mu,T_1)+i_e(q|\mu,T_2). \tag{5.2}$$

From this, a standard argument yields that $i_e(q|\mu,T)/T\to i_e(q|\mu)$, which is the content of the assertion (2.19).

Step 2. Lower semicontinuity of $\boldsymbol{i_e(\cdot|\mu)}$. Fix $q_0,q>0$ and set $\lambda=q/q_0$. Given $(\boldsymbol{\pi},\boldsymbol{Q})\in\mathscr{A}_{e,T}(q|\mu)$, define

$$\tilde{\pi}^\lambda_t=\pi_{t/\lambda},\quad \tilde{Q}^\lambda(\mathrm{d}t)=\frac{1}{\lambda}Q(\mathrm{d}t/\lambda)$$

for $\lambda>0$. In particular, $(\tilde{\pi}^\lambda,\tilde{Q}^\lambda)\in\mathscr{A}_{\lambda T}(q_0|\mu)$. By direct computation,

$$\begin{aligned}J_{e,\lambda T}(\tilde{\pi}^\lambda,\tilde{Q}^\lambda) &= \int_0^{\lambda T}\tilde{Q}^\lambda(\mathrm{d}t)\log\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}Q^{\tilde{\pi}^\lambda}}-\tilde{Q}^\lambda(1)+Q^{\tilde{\pi}^\lambda}(1)\\ &=\lambda T q_0\log\frac{1}{\lambda}+(\lambda-1)Q^\pi(1)+J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q}),\end{aligned}$$

Since $Q^\pi(1)\le cT$, we find, for some constant c depending on e,

$$i_e(q_0|\mu)\le\frac{1}{\lambda}i_e(q|\mu)+\frac{q}{\lambda}\log\lambda+\left|1-\frac{1}{\lambda}\right|c=\frac{q_0}{q}i_e(q|\mu)+q_0\log\frac{q_0}{q}+\left|1-\frac{q_0}{q}\right|c,$$

Taking the limit inferior for $q\to q_0$ produces the lower-semicontinuit**y of** i_e**.**

Step 3. Independence of $\boldsymbol{i_e(q|\mu)}$ on $\boldsymbol{\mu}$. We first prove, for any $\mu\in\mathscr{P}_e$, the inequality

$$i_e(q|\mu)\le i_e(q|M_e). \tag{5.3}$$

Fix $T > 0$ and $(\boldsymbol{\pi}, \boldsymbol{Q}) \in \mathscr{A}_{e,T}(q|M_e)$, and let $(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{Q}}) \in \mathscr{S}_{e,1}$ be the path satisfying $\hat{\pi}_0 = \mu$, $\hat{\pi}_1 = M_e$, provided by Lemma 4.2. Set $\tilde{T} = T + 2$ and $(\tilde{\boldsymbol{\pi}}, \tilde{\boldsymbol{Q}}) \in S_{e,\tilde{T}}$ be defined by

$$\tilde{\pi}_t = \begin{cases} \hat{\pi}_t & \text{if } t \in [0,1) \\ \pi_{t-1} & \text{if } t \in [1, \tilde{T}-1] \\ \hat{\pi}_{\tilde{T}-t} & \text{if } t \in (\tilde{T}-1, \tilde{T}] \end{cases}$$

and

$$\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}t} = \begin{cases} \frac{\mathrm{d}\hat{Q}}{\mathrm{d}t}(t) & \text{if } t \in [0,1) \\ \frac{\mathrm{d}Q}{\mathrm{d}t}(t-1) & \text{if } t \in [1, \tilde{T}-1] \\ \frac{\mathrm{d}\hat{Q}}{\mathrm{d}t}(\tilde{T}-t) \circ \Upsilon & \text{if } t \in (\tilde{T}-1, \tilde{T}]. \end{cases}$$

Let $\tilde{q}_T = \frac{1}{\tilde{T}}\tilde{Q}(1)$. By construction, $|\tilde{q}_{\tilde{T}} - q| \le c/T$. By construction and Lemma 4.2,

$$i_e(\tilde{q}_T|M_e) \le \frac{1}{\tilde{T}} J_{e,\tilde{T}}(\tilde{\boldsymbol{\pi}}, \tilde{\boldsymbol{Q}}) \le \frac{1}{\tilde{T}} J_{e,T}(\boldsymbol{\pi}, \boldsymbol{Q}) + \frac{c}{T}.$$

In view of the subadditivity proven in step 1, and the lower semicontinuity proven in step 2, by optimizing over $(\boldsymbol{\pi}, \boldsymbol{Q}) \in \mathscr{A}_{e,T}(q|M_e)$ and taking the limit inferior as $T \to +\infty$, we deduce the inequality (5.3). The reverse inequality is proven by the same argument, and it follows for all $\mu, \nu \in \mathscr{P}_e$ and all $q > 0$,

$$i_e(q|\mu) = i_e(q|M_e) = i_e(q|\nu). \tag{5.4}$$

Step 4. Convexity on $(0, +\infty)$. Thanks to step 3, we take $\mu = M_e$ and omit it from the notation. Thanks to lower semicontinuity, it is sufficient to prove that, for each $q_1, q_2 \in (0, +\infty)$,

$$i_e\Big(\frac{q_1 + q_2}{2}\Big) \le \frac{i_e(q_1) + i_e(q_2)}{2}. \tag{5.5}$$

Fix μ, let T_n^1 be any diverging sequence, and let $(\pi_n^1, Q_n^1) \in \mathscr{A}_{e,T_n}(q_1|\mu)$ be chosen such that

$$\lim_{n\to+\infty} \frac{1}{T_n} J_{e,T_n}(\boldsymbol{\pi}_n^1, \boldsymbol{Q}_n^1) = i_e(q_1).$$

For $(\boldsymbol{\pi}_n, \boldsymbol{Q}_n) \in \mathscr{A}_{e,2T_n}((q_1+q_2)/2|\mu)$ as the path obtained by concatenating $(\boldsymbol{\pi}_n^1, \boldsymbol{Q}_n^1)$ and $(\boldsymbol{\pi}_n^2, \boldsymbol{Q}_n^2)$, then

$$i_e\Big(\frac{q_1+q_2}{2}|\mu\Big) \le \frac{1}{2T_n}\Big(J_{e,T_n}(\boldsymbol{\pi}_n^1, \boldsymbol{Q}_n^1) + J_{e,T_n}(\boldsymbol{\pi}_n^2, \boldsymbol{Q}_n^2)\Big).$$

We deduce (5.5) by taking the limit $n \to +\infty$.

Step 5. i_e is continuous on $(0, +\infty)$. By convexity, it is enough to show that i_e is bounded. Recalling that $i_e(q) = i_e(q|M_e)$, by choosing $\pi = M_e$ and $Q = \alpha Q^{M_e}$ with $\alpha = q/\bar{q}_e$ and $\bar{q}_e$ defined in (2.15), we obtain that

$$i_e(q) \le q \log q/\bar{q}_e - (q - \bar{q}_e) < +\infty.$$

Step 6. $i_e(q) = 0$ for $q \in (0, \bar{q}_e]$. Given $q \in (0, \bar{q}_e)$, let e' be such that $\bar{q}_{e'} = q$, where $\bar{q}_e$ is defined in (2.15). Observe that $i_e(q) = i_e(q|M_{e'})$. By choosing the path $(M_{e'}, Q^{M_{e'}})$, which is in $\mathscr{A}_{e,T}(q|M_{e'})$ for any $T > 0$, we obtain $i_e(q) = 0$. This statement gives also the convexity and continuity of i_e in $[0, +\infty)$. ∎

Proof of Theorem 2.3. We collect the proofs of the different parts of equicoercivity and Γ-convergence.

Proof of i) (Equicoercivity). Fix $q>0$. For any $T>0$ and any path $(\boldsymbol{\pi},\boldsymbol{Q})$ in $[0,T]$, such that $\boldsymbol{Q}(1)=Tq$, by choosing the constant function $\gamma>0$ in the variational formula for $J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q})$ proven in [1],

$$\frac{1}{T}J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q})\geq q\gamma-(\mathrm{e}^{\gamma}-1)\frac{1}{T}\boldsymbol{Q}^{\pi}(1)\geq q\gamma-(\mathrm{e}^{\gamma}-1)C,$$

where C is a constant depending only on e. The statement follows.

Proof of ii) (Γ–lim inf). Fix $q\in[0,+\infty)$ and consider a sequence $q_T\to q$ as $T\to+\infty$. Recalling the definition (2.16) of $\mathscr{J}_{e,T}$,

$$\frac{1}{T}\mathscr{I}_{e,T}(q_T|m)=\inf_{(\boldsymbol{\pi},\boldsymbol{Q}):\boldsymbol{Q}(1)=Tq_T}\left\{\frac{1}{T}H_e(\pi_0|m)+\frac{1}{T}J_{e,T}(\boldsymbol{\pi},\boldsymbol{Q})\right\}.$$

By the goodness of the rate function (2.13), the infimum is achieved for some path (π^T,Q^T) with $Q^T(1)=Tq_T$. For any $T_0>0$, the Controllability Theorem 4.1 produces a path $(\bar{\boldsymbol{\pi}},\bar{\boldsymbol{Q}})\in\mathscr{S}_{e,T_0}$ with $\bar\pi_0=\pi^T_T$, $\bar\pi_{T_0}=\pi^T_0$ and, provided T_0 is chosen so that $qT_0\geq C_1(e)$, $\bar{\boldsymbol{Q}}(1)=qT_0$, satisfying

$$J_{e,T_0}(\bar{\boldsymbol{\pi}},\bar{\boldsymbol{Q}})\leq F_1(e,T_0,qT_0)+H_e(\pi^T_0|M_e).$$

We now let $(\tilde{\boldsymbol{\pi}},\tilde{\boldsymbol{Q}})$ be the path in $[0,T+T_0]$ given by concatenating $(\bar{\boldsymbol{\pi}},\bar{\boldsymbol{Q}})$ and $(\boldsymbol{\pi}^T,\boldsymbol{Q}^T)$. This produces

$$J_{e,T+T_0}(\tilde{\boldsymbol{\pi}},\tilde{\boldsymbol{Q}})\leq F_1(e,T_0,qT_0)+H_e(\pi^T(0)|M_e)+J_{e,T}(\boldsymbol{\pi}^T,\boldsymbol{Q}^T).$$

By assumption 2.1, there exists a constant depending only on m and e such that

$$H_e(\pi^T(0)|M_e)\leq c+H_e(\pi^T(0)|m).$$

Then,

$$\frac{1}{T+T_0}J_{e,T+T_0}(\tilde{\boldsymbol{\pi}},\tilde{\boldsymbol{Q}})\leq\frac{c+F_1(e,T_0,qT_0)}{T+T_0}+\frac{T}{T+T_0}\frac{1}{T}\mathscr{I}_{e,T}(q_T|m).$$

Note that $(\tilde\pi,\tilde Q)\in\mathscr{A}_{e,T+T_0}(\tilde q_T,\pi^T_0)$ for some $\tilde q_T\to q$ as $T\to+\infty$. Recalling (2.17), we conclude by taking the lim inf and using the lower semicontinuity of $i_e(q)$ proven in Proposition 2.2.

Proof of iii) (Γ–lim sup). We split into the cases $q>0,q=0$.
For $q>0$, we apply Proposition 2.2 to see that there exists a sequence of paths $(\boldsymbol{\pi}^T,\boldsymbol{Q}^T)$, such that $\pi^T(0)=m=\pi^T(T)$, $\boldsymbol{Q}^T(1)=qT$ and

$$\lim_{T\to+\infty}\frac{1}{T}J_{e,T}(\boldsymbol{\pi}^T,\boldsymbol{Q}^T)=i_e(q).$$

By choosing the constant sequence $q_T=q$, we deduce

$$\mathscr{I}_{e,T}(q|m)\leq\frac{1}{T}\{H_e(\pi^T(0)|m)+J_{e,T}(\boldsymbol{\pi}^T,\boldsymbol{Q}^T)\},$$

which yields the statement.

For $q=0$, recalling the separate definition of $i_e(0)$ in (2.18), there exists a sequence $q_n\downarrow 0$ such that $i_e(0)=\lim_n i_e(q_n)$. By the previous result, using a diagonal argument, we conclude the proof also for $q=0$. ∎

Proof of Theorem 2.5.

Upper bound. We first prove that $i_e \le i_e^+$ given by (2.20). Given $\mu \in \mathscr{P}_e$ with density f, consider the path $(\boldsymbol{\pi}, \boldsymbol{Q})$ with $\pi_t = \mu$ and $\mathrm{d}Q = \mathrm{e}^\gamma \frac{1}{2}\sqrt{ff_*f'f'_*}B\,\mathrm{d}t\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega$. By the symmetry of Q, $(\boldsymbol{\pi}, \boldsymbol{Q})$ satisfies the continuity equation (2.8) and, tuning γ so that $\mathrm{e}^\gamma = q/R_4(\mu)$, we achieve $(\boldsymbol{\pi}, \boldsymbol{Q}) \in \mathscr{A}_{e,T}(q|\mu)$. By direct computation,

$$\frac{1}{T}J_{e,T}(\boldsymbol{\pi}, \boldsymbol{Q}) = q\log\frac{q}{R_4(\mu)} - q + R_2(\mu).$$

We conclude by optimizing in μ.

Lower bound. We now prove the lower bound given by (2.21).

By the variational representation of $J_{e,T}$ proven in [1], for each $F\colon [0,T]\times\mathbb{R}^{2d}\times\mathbb{R}^{2d}\to\mathbb{R}$ continuous, bounded, and satisfying $F(t;v,v_*,v',v'_*) = F(t;v_*,v,v',v'_*) = F(t;v,v_*,v'_*,v')$,

$$\frac{1}{T}J_{e,T}(\boldsymbol{\pi}, \boldsymbol{Q}) \ge \frac{1}{T}\left(\boldsymbol{Q}(F) - \boldsymbol{Q}^\pi(\mathrm{e}^F - 1)\right).$$

Given a path $(\boldsymbol{\pi}, \boldsymbol{Q}) \in \mathscr{A}_{e,T}(q|\mu)$, by choosing $F = \gamma + \frac{1}{2}\log\frac{f'f'_*}{ff_*}$, where f is the density of $\pi(t)$, we deduce

$$\frac{1}{T}J_{e,T}(\boldsymbol{\pi}, \boldsymbol{Q}) \ge \gamma q - \mathrm{e}^\gamma\frac{1}{T}\int_0^T R_4(\pi_t)\,\mathrm{d}t + \frac{1}{T}\int_0^T R_2(\pi_t)\,\mathrm{d}t,$$

where we use $\pi_0 = \pi_T$, so that, by the chain rule (3.4), $Q(\log\frac{f'f'_*}{ff_*}) = 0$. Optimizing in γ, we have

$$\frac{1}{T}J_{e,T}(\boldsymbol{\pi}, \boldsymbol{Q}) \ge q\log\frac{q}{\frac{1}{T}\int_0^T R_4(\pi_t)\,\mathrm{d}t} - q + \frac{1}{T}\int_0^T R_2(\pi_t)\,\mathrm{d}t.$$

Since $R_2(\pi_t) \ge R_4(\pi_t)$ and $R_4(\pi_t) \le \hat{q}_e$, we deduce

$$\frac{1}{T}J_{e,T}(\boldsymbol{\pi}, \boldsymbol{Q}) \ge \inf_{0<a<\hat{q}_e}\left(q\log\frac{q}{a} - q + a\right) = i_e^-(q). \qquad \square$$

6. Second order asymptotics. We finally turn to the proof of Theorem 2.6 concerning the second-order asymptotics on the interval $[0, \bar{q}_e]$. The proof is based on a Young measure argument: Informally, given a path $(\boldsymbol{\pi}^T, \boldsymbol{Q}^T)$ of length T, we associate with it a time average θ_T by (6.1), which is a probability measure on pairs of measures and fluxes. The assertions of the Theorem will follow by identifying properties of limit points $\theta_T \to \theta, T\to\infty$ for any sequence $(\boldsymbol{\pi}^T, \boldsymbol{Q}^T)$ along which the rate $I_{e,T}$ remains bounded.

Before turning to the proof, we formalize this discussion. Denote by $\mathscr{M}$ the subset of the finite measures $\boldsymbol{Q}$ on $\mathbb{R}^{2d}\times\mathbb{R}^{2d}$ that satisfy $\boldsymbol{Q}(\mathrm{d}v,\mathrm{d}v_*,\mathrm{d}v',\mathrm{d}v'_*) = \boldsymbol{Q}(\mathrm{d}v_*,\mathrm{d}v,\mathrm{d}v',\mathrm{d}v'_*) = \boldsymbol{Q}(\mathrm{d}v,\mathrm{d}v_*,\mathrm{d}v'_*,\mathrm{d}v')$. Given $\pi\in\mathscr{P}_e$, set

$$\boldsymbol{Q}^\pi = \frac{1}{2}\pi(\mathrm{d}v)\pi(\mathrm{d}v_*)B(v-v_*,\mathrm{d}\omega) \in \mathscr{M}.$$

Fix $q\in[0,+\infty)$ and a sequence $q_T\to q$ as $T\to\infty$. Fix also a sequence $(\boldsymbol{\pi}^T, \boldsymbol{Q}^T) \in \mathscr{S}_{e,T}$ such that $\boldsymbol{Q}^T(1) = Tq_T$, and

$$\limsup_{T\to+\infty} I_{e,T}((\boldsymbol{\pi}^T, \boldsymbol{Q}^T)|M_e) < +\infty.$$

Each $\boldsymbol{Q}^T$ may be written as $\boldsymbol{Q}^T(\mathrm{d}t) = \mathrm{d}t\, Q_t^T$, with $Q_t^T \in \mathscr{M}$, a.e. in $t \in [0,T]$. Let us introduce the time average associated to the $\boldsymbol{Q}^T$ given by

$$\vartheta_T = \frac{1}{T}\int_0^T \mathrm{d}t\, \delta_{\pi_t^T, Q_t^T}, \tag{6.1}$$

which is a probability measure on $\mathscr{P}_e \times \mathscr{M}$.

LEMMA 6.1. *Given the hypotheses above on* $(\boldsymbol{\pi}^T, \boldsymbol{Q}^T)$, *the sequence* $(\vartheta_T)_{T>0}$ *is precompact. Furthermore, if* ϑ *is any cluster point of* ϑ_T, *then*

i) $\int \vartheta(\mathrm{d}\pi, \mathrm{d}Q)\, Q(1) = q$,

ii) *for* ϑ *a.e.* (π, Q), *it holds that* $Q = Q^\pi = Q \circ \Upsilon$. *In particular,* ϑ *is supported on the set*

$$\mathcal{G}_e := \{(M_{e'}, Q^{M_{e'}}) : 0 \le e' \le e\}$$

of pairs consisting of Maxwellian of energy at most e *and its associated measures* $Q^{M_{e'}}$.

Proof. We start by proving the compactness. Since $\mathscr{P}_e$ is compact, by Chebyshev's inequality and Prohorov's theorem, it is enough to show that there exists $\Phi : \mathbb{R}^{4d} \to [0, +\infty)$ with compact level sets such that

$$\limsup_{T\to+\infty} \int \vartheta_T(\mathrm{d}\pi, \mathrm{d}Q)\, Q(\Phi) < +\infty. \tag{6.2}$$

Choosing $\Phi = \frac{1}{2}\log(1 + |v|^2 + |v_*|^2 + |v'|^2 + |v'_*|^2)$, the condition $\pi \in \mathscr{P}_e$ implies

$$\sup_{\pi\in\mathscr{P}_e} Q^\pi(\mathrm{e}^\Phi) < +\infty.$$

For any $Q \ll Q^\pi$, by the Legendre duality

$$Q(\Phi) \le Q^\pi(\mathrm{e}^\Phi - 1) + \int \mathrm{d}Q^\pi \left(\frac{\mathrm{d}Q}{\mathrm{d}Q^\pi} \log \frac{\mathrm{d}Q}{\mathrm{d}Q^\pi} - \frac{\mathrm{d}Q}{\mathrm{d}Q^\pi} + 1 \right),$$

so that

$$\int \vartheta_T(\mathrm{d}\pi, \mathrm{d}Q)\, Q(\Phi) \le \sup_{\pi\in\mathscr{P}_e} Q^\pi(\mathrm{e}^\Phi) + \frac{1}{T} J_{e,T}(\boldsymbol{\pi}^T, \boldsymbol{Q}^T),$$

which concludes the proof of (6.2).

Let ϑ be a cluster point of ϑ_T and pick a sequence of T such that $\vartheta_T \to \vartheta$. By definition of ϑ_T,

$$\int \vartheta_T(\mathrm{d}\pi, \mathrm{d}Q) Q(1) = q_T.$$

Item *si)* follows from this identity and the uniform integrability given by (6.2).

By definition,

$$\int \vartheta_T(\mathrm{d}\pi, \mathrm{d}Q) \int \mathrm{d}Q^\pi \left(\frac{\mathrm{d}Q}{\mathrm{d}Q^\pi} \log \frac{\mathrm{d}Q}{\mathrm{d}Q^\pi} - \frac{\mathrm{d}Q}{\mathrm{d}Q^\pi} + 1 \right) \le \frac{1}{T} I_{e,T}((\boldsymbol{\pi}^T, \boldsymbol{Q}^T)|M_e).$$

Using Fatou's lemma to bound the limit of the left-hand side, and since the rate function appearing on the right-hand side is finite by the hypothesis,

$$\int \vartheta(\mathrm{d}\pi,\mathrm{d}Q)\int \mathrm{d}Q^\pi\left(\frac{\mathrm{d}Q}{\mathrm{d}Q^\pi}\log\frac{\mathrm{d}Q}{\mathrm{d}Q^\pi}-\frac{\mathrm{d}Q}{\mathrm{d}Q^\pi}+1\right)=0,$$

which implies that ϑ a.e. $Q=Q^\pi$. By Proposition 3.1,

$$\limsup_{T\to+\infty} I(\boldsymbol{\pi}^T,\boldsymbol{Q}^T\circ\Upsilon)<+\infty.$$

Arguing as before, we deduce that ϑ a.e. $Q\circ\Upsilon=Q^\pi$. As follows from [10, §3.2], the probability measures π satisfying $Q^\pi=\Upsilon\circ Q^\pi$ are the Maxwellians $M_{e'}:e'\geq 0$. Since the finiteness of $I_{e,T}$ imposes that $\pi_t^T(\zeta_0)\leq e'$ for all t, it follows by lower semicontinuity that $\pi(\zeta_0)\leq e$ for ϑ-almost all (π,Q). Together, this proves that ϑ is supported on $\mathcal{G}_e$, as claimed. ∎

Proof of Theorem 2.6. The equi-coercivity in item (i) follows the analogous statement in Theorem 2.3.

Proof of ii) (Γ–lim inf). In this step, we show that, for any sequence $q_T\to q\in[0,\infty)$, it holds that

$$\liminf_{T\to\infty}\mathcal{J}_{e,T}(q_T|M_e)\geq j_e(q), \tag{6.3}$$

where j_e is defined by (2.22). We divide it into the cases where $q>\bar q_e, q\leq \bar q_e$.

For $q>\bar q_e$, that the $\Gamma-$liminf in (ii) is infinite for $q>\bar q_e$ would follow from the first order asymptotic if we had proven that $i_e(q)>0$ for $q>\bar q_e$. Since we have proved it only for q large enough, we need a separate argument. Fix $q>\bar q_e$ and suppose that $q_T\to q$, let us assume for a contradiction that

$$\liminf_{T\to+\infty}\mathscr{I}_{e,T}(q_T|M_e)<+\infty. \tag{6.4}$$

We may therefore choose competitors $(\boldsymbol{\pi}^T,\boldsymbol{Q}^T)\in\mathcal{S}_{e,T}$ with $\boldsymbol{Q}^T(1)=Tq_T$ and

$$I_{e,T}((\boldsymbol{\pi}^T,\boldsymbol{Q}^T)|M_e)<1+\mathscr{I}_{e,T}(q_T|M_e).$$

It therefore follows that the liminf of the left-hand side is finite as $T\to\infty$, and we are in the setting of Lemma 6.1. As a result, there exists a sequence $T_n\to\infty$ along which the time averages ϑ_{T_n} given by (6.1) converge to some ϑ, satisfying $\int\vartheta(d\pi,dQ)Q(1)=q$ and with support in $\mathscr{G}_e$. On the other hand, the support condition implies that $\int\vartheta(\mathrm{d}\pi,\mathrm{d}Q)\,Q(1)\leq\bar q_e$, which provides a contradiction. We conclude that the original hypothesis (6.4) is false. As a result, the liminf appearing in (6.4) is infinite for all $q>\bar q_e$, which is the conclusion (6.3) in the case $q>\bar q_e$.

We next consider the case where $\lim_{T\to+\infty}q_T=q\leq\bar q_e$. If there is no sequence along which $\mathscr{I}_{e,T}(q_T|M_e)$ is bounded, then the conclusion is trivial. Otherwise, we may pick a subsequence and competitors $(\boldsymbol{\pi}^T,\boldsymbol{Q}^T)$ satisfying $\boldsymbol{Q}^T(1)=q_T$ and for which $I_{e,T}(\boldsymbol{\pi}^T,\boldsymbol{Q}^T|M_e)$ is bounded. Taking the average ϑ_T as in (6.1), we may pass to a further subsequence on which $\vartheta_T\to\vartheta$ for some ϑ. Since $\int\vartheta(\mathrm{d}\pi,\mathrm{d}Q)\,Q(1)=q$, ϑ gives positive probability to $\{(M_{e'},Q^{M_{e'}}),\,e'\in[0,\epsilon(q)]\}$. As a consequence, there exists a sequence $t=t(T)\leq T$, $t\uparrow+\infty$ such that $\boldsymbol{\pi}_t^T\to M_{\tilde e}$ with $\tilde e\leq\epsilon(q)$. By Proposition 3.1,

$$I_{e,T}((\boldsymbol{\pi}^T,\boldsymbol{Q}^T)|M_e)\geq H_e(\boldsymbol{\pi}_t^T).$$

By lower semicontinuity of H_e,

$$\liminf_{T\to+\infty} H_e(\boldsymbol{\pi}_t^T) \geq H_e(M_{\tilde{e}}) \geq H_e(M_{\epsilon(q)}) = j_e(q)$$

and the claim (6.3) is proven for $q \leq \bar{q}_e$.

Proof of iii) (Γ-lim sup). The Γ-lim sup in (iii) is trivial if $q > \bar{q}_e$. For $q \leq \bar{q}_e$ it is enough to choose $q_T = q$, and the path $\boldsymbol{\pi}_t^T = M_{\varepsilon(q)}$, $\boldsymbol{Q}^T = \boldsymbol{Q}^{\boldsymbol{\pi}^T}$.

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